Discrete and Continuous Dynamical Systems with Applications

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Contents

Introduction

These lecture notes contain the material of the course Discrete and continuous dynamical systems offered for master students at the Faculty of Information Technology, University of Pannonia. The goal of this course is to give a short introduction to some basic topics of the theory of differential and difference equations.

The organization of the lecture notes is the following. Chapter 1 studies some basic notions and terminology of differential equations and some solution techniques and applications for first-order scalar differential equations. Chapter 2 discusses the theory of second-order linear equations and classical applications including spring-mass system and the pendulum. Chapter 3 contains the basic methods of linear homogeneous systems of differential equations together with some necessary background from linear algebra. Chapter 4 gives an introduction to stability notions and results for differential equations, and in the related Chapter 5 we introduce some notions from the theory of bifurcations. Chapter 6 includes a few examples for the case when time delay appears naturally in differential equation models. Chapter 7 shows some basic solution techniques of first-order linear difference equations through discrete population models. Chapter 8 discusses solution techniques of higher-order difference equations, and Chapter 9 presents some basic results and definitions for stability and bifurcation theory for difference equations. Finally, Chapter 10 shows an application when the model is a so-called hybrid system, a combination of continuous and discrete system.

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Chapter 1
First-order differential equations

In this chapter, we introduce some basic notions and notations for first order differential equations, study some solutions techniques for linear and for simple classes of nonlinear first-order differential equations. We present some applications where first-order differential equations are used as model equations. We also summarize basic existence and uniqueness results for scalar and vector valued first-order differential equations.

1.1 Basic concepts

Differential equation is an equation which involves an unknown function and its derivatives. An *ordinary differential equation* is a differential equation where the unknown is a function of a single variable, a *partial differential equation* is a differential equation where the unknown function is of several variables. In these lecture notes, we study only ordinary differential equations (ODEs). An ODE is said to be of order n if the nth derivative of the unknown function appears in the equation but no larger order derivatives. An nth-order ODE has the general form

$$g(x, y, y', \dots, y^{(n-1)}, y^{(n)}) = 0. \tag{1.1}$$

Here y is the unknown function of x, i.e., $y = y(x)$. It is usual in differential equations to omit the argument of the unknown function in the equation. We will use this convention throughout these lecture notes.

In Eq. (1.1) all terms are moved to the left-hand-side of the equation. Then the left-hand-side can be any expression of the independent variable x, the unknown function y and its derivatives up to order n. In a particular case, of course, any term except $y^{(n)}$ can be omitted in the equation. We note that in many applications, the independent variable is the time. Therefore in many cases t is used to denote the independent variable of the unknown function in the ODE instead of x.

Eq. (1.1) is called *nth-order implicit differential equation*. In the case when the equation can be solved for the largest order derivative, we can rewrite the differential equation in the form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \tag{1.2}$$

which is called *explicit nth-order ODE*. By a *solution* of Eq. (1.2) we mean a function y which is defined on an interval $I \subset \mathbb{R}$, and which satisfies (1.2) for $x \in I$. Therefore a solution is defined on an interval by definition. No restriction is given whether this interval is finite or infinite, open or closed.

Example 1.1 The equation

$$4x^3y^{(4)} - 5y(y'')^2 + 7x^2 = 0$$

is a fourth-order implicit ODE which can be rewritten as an explicit fourth-order ODE of the form

$$y^{(4)} = \frac{5y(y'')^2 - 7x^2}{4x^3}.$$

We note that the two equations are not equivalent, since the first equation is defined on $\mathbb{R}$, but the second equation is defined only for $x \neq 0$. □

Example 1.2 The equation

$$y'' = 2x + 1$$

is a very simple explicit second-order ODE which can be solved by integrating both sides of the equation twice. First, integrating by x we get

$$y' = \int (2x + 1)\,\mathrm{d}x,$$

hence

$$y' = x^2 + x + c_1.$$

Integrating once more we get

$$y = \int (x^2 + x + c_1)\,\mathrm{d}x,$$

so

$$y = \frac{x^3}{3} + \frac{x^2}{2} + c_1 x + c_2.$$

This formula gives all solutions of the original ODE, since we made equivalent transformations. The formula of the solution contains two independent constants c_1 and c_2, which can take any real value. □

The previous example illustrates that an ODE has infinitely many solutions. A formula which contains n independent constants (parameters) is called a *general solution* of the nth-order equation (1.1) or (1.2) if the formula satisfies the equation on an interval for any selection of the parameters form the domain of the parameters. So the formula given at the end of the previous example gives the general solution of the ODE. It can happen that a general solution of an ODE does not contain all solutions. Such a solution which can not be obtained from the formula of the general solution by fixing the parameter values is called *singular solution*.

Given an nth-order ODE and its general solution containing n parameters, then, in general, n predefined conditions are needed to guarantee the uniqueness of the solution. The most frequently used conditions are of the form

$$y(x_0) = z_1, \quad y'(x_0) = z_2, \quad \cdots, \quad y^{(n-1)}(x_0) = z_n. \tag{1.3}$$

Conditions (1.3) are called *initial conditions* (IC), x_0 is called *initial time*, the given $z_1, z_2, \ldots, z_n$ values are called *initial values*. Eq. (1.1) (or (1.2)) together with the IC (1.3) is called *initial value problem* (IVP).

Example 1.3 Consider the first-order ODE

$$(\mathrm{e}^{3y} + 8y)y' = 7x^5.$$

One can recognize that the left-hand-side of this equation is a derivative of a composite function, so it can be rewritten as

$$\left(\frac{1}{3}\mathrm{e}^{3y} + 4y^2\right)' = 7x^5.$$

Integrating both sides we get

$$\frac{1}{3}\mathrm{e}^{3y} + 4y^2 = \frac{7x^6}{6} + c.$$

This nonlinear equation is called the *implicit solution* of the ODE. From this equation, we cannot express y as a function of x, so we can not give the *explicit solution* of the ODE. But using numerical methods for any given x, we can find the approximate solution $y(x)$ of the nonlinear equation with arbitrary precision. □

1.2 Separable differential equations

The first-order scalar ODE

$$y' = g(x)h(y) \tag{1.1}$$

is called *separable differential equation*. The method of solving Eq. (1.1) is the following. By division or multiplication we *separate the variables*, i.e., the same variables are rearranged to the same side of the equation:

$$\frac{y'}{h(y)} = g(x),$$

then we integrate both sides of the equation by x. To see the details of the calculations here we write out the argument of y:

$$\int \frac{y'(x)}{h(y(x))}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x.$$

The function y in the left integral is not known, but we can compute this integral if we use the change of variable $u = y(x)$: using the formal computation rule $du = y'(x)\,dx$ we get

$$\int \frac{1}{h(u)}\,\mathrm{d}u = \int g(x)\,\mathrm{d}x. \tag{1.2}$$

Computing both integrals and substituting u with y, we get the implicit solution of the equation.

The next example illustrates the above method.

Example 1.4 Consider the first-order scalar ODE

$$y' = (3x^4 + 2x)y^2. \tag{1.3}$$

This is a separable differential equation, since by division we can separate the variables:

$$\frac{y'}{y^2} = 3x^4 + 2x.$$

Here the left-hand-side depends only on y, and the right-hand-side only on x. Integrating both sides with respect to x gives

$$\int \frac{y'(x)}{y^2(x)}\,\mathrm{d}x = \int (3x^4 + 2x)\,\mathrm{d}x.$$

Using the new variable $u = y(x)$ we get

$$\int \frac{1}{u^2}\,\mathrm{d}u = \int (3x^4 + 2x)\,\mathrm{d}x,$$

and computing the indefinite integrals

$$-\frac{1}{u} + c_1 = \frac{3x^5}{5} + x^2 + c_2.$$

We can introduce $c = c_2 - c_1$ instead of the two independent constants c_1 and c_2, so

$$-\frac{1}{u} = \frac{3x^5}{5} + x^2 + c.$$

We can conclude that whenever we integrate both sides of an equation it is always enough to write $+c$ only in one side of the equation. Replacing back u by y and omitting the argument of y as usual, we get that

$$-\frac{1}{y} = \frac{3x^5}{5} + x^2 + c$$

is the implicit solution of the equation. Here we can solve this algebraic equation for y, so

$$y = -\frac{1}{3x^5/5 + x^2 + c}$$

is the explicit solution of the ODE. □

Now consider again Eq. (1.1). We give a practical computation method as follows. First replace y' by the classical notation $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. This yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y). \tag{1.4}$$

Consider the left-hand-side formally as a fraction, and now separate the variables:

$$\frac{\mathrm{d}y}{h(y)} = g(x)\,\mathrm{d}x.$$

We have to emphasize that this is a formal equation. To give a meaning of this equation, we "integrate" both sides, i.e., write integral sign on both sides:

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x.$$

Now on the left-hand-side y is used as an independent variable, and the formula denotes an integration with respect to y, on the right-hand-side an integration with respect to x. Note that this is equivalent to Eq. (1.2), only there instead of the variable y variable u was used, which in the next step was substituted with y.

We solve again Example 1.4 using this formal calculation.

Example 1.5 Consider again Eq. (1.3), but using $\frac{\mathrm{d}y}{\mathrm{d}x}$ notation for y':

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x^4 + 2x)y^2.$$

Separating the variables gives

$$\frac{\mathrm{d}y}{y^2} = (3x^4 + 2x)\,\mathrm{d}x,$$

then we "integrate" both sides

$$\int \frac{\mathrm{d}y}{y^2} = \int (3x^4 + 2x)\,\mathrm{d}x.$$

Computing both integrals yields the same solution that we got in Example 1.4 with a longer calculation:

$$-\frac{1}{y} = \frac{3x^5}{5} + x^2 + c.$$

□

Now we show a third possible method for solving the separable ODE (1.1). Suppose given an IC

$$y(x_0) = y_0 \tag{1.5}$$

associated to Eq. (1.1). As in the first two methods we separate the variables in Eq. (1.1)

$$\frac{y'(x)}{h(y(x))} = g(x),$$

but now we compute the definite integrals of both sides from x_0 to x (which is one possible antiderivative of the itegrands). Since x will be the upper limit of the integrals, we use a different arguments inside the integrals:

$$\int_{x_0}^{x} \frac{y'(t)}{h(y(t))}\,\mathrm{d}t = \int_{x_0}^{x} g(t)\,\mathrm{d}t.$$

In the first integral we substitute the new variable $u = y(t)$, so we get

$$\int_{y(x_0)}^{y(x)} \frac{1}{h(u)}\,\mathrm{d}u = \int_{x_0}^{x} g(t)\,\mathrm{d}t,$$

hence, using the IC, we get

$$\int_{y_0}^{y} \frac{1}{h(u)}\,\mathrm{d}u = \int_{x_0}^{x} g(t)\,\mathrm{d}t.$$

Note that in the upper limit of the first integral $y(x)$ is simply written as y, as we usually do in differential equations.

The next example illustrates the above calculation:

Example 1.6 Consider the IVP

$$y' = \frac{\mathrm{e}^{x^2}}{3y^3}, \qquad y(1) = 2.$$

Separating the variables (using formal calculation) gives

$$3y^3\mathrm{d}y = \mathrm{e}^{x^2}\mathrm{d}x.$$

Computing the indefinite integrals of both sides yields

$$\int 3y^3\,\mathrm{d}y = \int \mathrm{e}^{x^2}\,\mathrm{d}x.$$

Now the problem is that the second integral has no antiderivative in terms of an elementary function, so we cannot compute this integral. Of course, we can give the answer in the implicit form

$$\frac{3y^4}{4} = \int \mathrm{e}^{x^2}\,\mathrm{d}x.$$

This is a general solution of the equation since the indefinite integral contains a parameter c implicitly. But in this form we cannot apply the IC, so we cannot give the solution of the IVP.

Now use definite integrals instead of the indefinite integrals above. Then using the IC we get

$$\int_2^y 3t^3\,\mathrm{d}t = \int_1^x \mathrm{e}^{t^2}\,\mathrm{d}t,$$

which implies

$$\frac{3y^4}{4} - \frac{3\cdot 2^4}{4} = \int_1^x \mathrm{e}^{t^2}\,\mathrm{d}t.$$

Here we give an explicit solution to the IVP:

$$y = \pm\sqrt[4]{16 + \frac{4}{3}\int_1^x \mathrm{e}^{t^2}\,\mathrm{d}t}.$$

Note that the given initial value at the initial time is positive, so the solution of this IVP is given by

$$y = \sqrt[4]{16 + \frac{4}{3}\int_1^x \mathrm{e}^{t^2}\,\mathrm{d}t}.$$

The solution is not expressed in terms of elementary functions, but using numerical methods we can evaluate the definite integral in the formula for any given x with arbitrary precision. This allows to use this formula in practice as it was a formula of elementary functions.

We note that if the initial value was $y(1) = -2$ then the solution of the IVP would be

$$y = -\sqrt[4]{16 + \frac{4}{3}\int_1^x \mathrm{e}^{t^2}\,\mathrm{d}t}.$$

□

1.3 First-order scalar linear differential equations

Let $I \subset \mathbb{R}$ be an open interval. An equation of the form

$$a(x)y' + b(x)y = g(x), \qquad x \in I \tag{1.1}$$

is called *first-order scalar linear differential equation*. In the case when $g \equiv 0$, the equation is called *homogeneous*, otherwise, i.e., when $g \not\equiv 0$, it is called *inhomogeneous* or *nonhomogeneous*. Therefore the general form of a scalar linear homogeneous differential equation is

$$a(x)y' + b(x)y = 0, \qquad x \in I. \tag{1.2}$$

Theorem 1.7 *Let y_1 and y_2 be solutions of the linear homogeneous differential equation (1.2) on the interval I. Then $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of (1.2) on I for any $\alpha_1, \alpha_2 \in \mathbb{R}$.*

Proof: Substitute the function $y = \alpha_1 y_1 + \alpha_2 y_2$ to the left-hand-side of (1.2). Then for $x \in I$

$$\begin{aligned} a(x)(\alpha_1 y_1 + \alpha_2 y_2)' + b(x)(\alpha_1 y_1 + \alpha_2 y_2) &= a(x)\alpha_1 y_1' + a(x)\alpha_2 y_2' + b(x)\alpha_1 y_1 + b(x)\alpha_2 y_2 \\ &= \alpha_1\Big(a(x)y_1' + b(x)y_1\Big) + \alpha_2\Big(a(x)y_2' + b(x)y_2\Big) \\ &= 0. \end{aligned}$$

□

Corollary 1.8 *The solutions of a first-order linear homogeneous equation form a linear space (vector space) of functions.*

Example 1.9 Consider the linear homogeneous equation

$$y' + 3x^2 y = 0.$$

Recognize that this equation is separable:

$$\frac{\mathrm{d}y}{y} = -3x^2\,\mathrm{d}x,$$

so integrating both sides we get

$$\ln|y| = -x^3 + C,$$

hence

$$y = \pm\mathrm{e}^{-x^3+C} = c\mathrm{e}^{-x^3},$$

where $c = \pm\mathrm{e}^C$. We can see that this formula gives a solution for $c = 0$ too, since $y \equiv 0$ satisfies the original equation. Note that we lost this solution in the above calculation when we divided the original equation by y. □

The above method can be applied in general. Consider the explicit form of Eq. (1.2), i.e., suppose $a(x) \neq 0$ on the interval I. Then introducing $r(x) = \frac{b(x)}{a(x)}$ Eq. (1.2) can be rewritten in the form

$$y' + r(x)y = 0, \qquad x \in I. \tag{1.3}$$

The solution of this equation is given in the next theorem.

Theorem 1.10 *The general solution of the first-order scalar linear homogeneous differential equation (1.3) is*

$$y_H = c\mathrm{e}^{-\int r(x)\,\mathrm{d}x}, \qquad x \in I, \quad c \in \mathbb{R}. \tag{1.4}$$

Proof: Eq. (1.3) is separable, so rewrite it as

$$\frac{\mathrm{d}y}{y} = -r(x)\,\mathrm{d}x.$$

Integrating both sides gives

$$\ln|y| = -\int r(x)\,\mathrm{d}x + C.$$

We note that writing $+C$ on the right-hand-side is superfluous, since it is included in the indefinite integral, but we use it to see the following calculation. We apply the exponential function to both sides

$$|y| = \mathrm{e}^{-\int r(x)\,\mathrm{d}x + C} = \mathrm{e}^C \mathrm{e}^{-\int r(x)\,\mathrm{d}x}.$$

Then $c = \pm\mathrm{e}^C$ gives (1.4), but the above calculation yields $c \neq 0$. On the other hand, if $c = 0$ then (1.4) gives $y \equiv 0$, which is also a solution of Eq. (1.3). □

Relation (1.4) implies that the space of the solutions is one-dimensional.

Corollary 1.11 *The solutions of a first-order linear homogeneous equation form a one-dimensional linear space.*

Theorem 1.12 *Let y_1 and y_2 be solutions of the first-order linear inhomogeneous equation (1.1) on the interval I. Then $y_1 - y_2$ is a solution of the linear homogeneous equation (1.2) on I.*

Proof: Substitute $y = y_1 - y_2$ to the left-hand-side of (1.1). Then for $x \in I$

$$\begin{aligned} a(x)(y_1 - y_2)' + b(x)(y_1 - y_2) &= a(x)y_1' - a(x)y_2' + b(x)y_1 - b(x)y_2 \\ &= \Big(a(x)y_1' + b(x)y_1\Big) - \Big(a(x)y_2' + b(x)y_2\Big) \\ &= g(x) - g(x) \\ &= 0. \end{aligned}$$

□

Theorem 1.13 *Let y_1 be a solution of the linear homogeneous equation (1.2), and y_2 be a solution of the linear inhomogeneous equation (1.1) on I. Then $y_1 + y_2$ is also a solution of the linear inhomogeneous equation (1.1) on I.*

Proof: Substitute $y = y_1 + y_2$ to the left-hand-side of (1.1). Then for $x \in I$

$$\begin{aligned} a(x)(y_1 + y_2)' + b(x)(y_1 + y_2) &= a(x)y_1' + a(x)y_2' + b(x)y_1 + b(x)y_2 \\ &= \Big(a(x)y_1' + b(x)y_1\Big) + \Big(a(x)y_2' + b(x)y_2\Big) \\ &= 0 + g(x) \\ &= g(x). \end{aligned}$$

□

A fixed solution of the linear inhomogeneous equation (1.1) is called a *particular solution*.

Corollary 1.14 *The general solution of the linear inhomogeneous equation (1.1) y_{IH} can be written as the sum of the general solution of the corresponding linear homogeneous equation y_H and a particular solution of the linear inhomogeneous equation y_{IP}:*

$$y_{IH} = y_H + y_{IP}.$$

In practice, for first-order linear inhomogeneous equations we use the so-called *method of integrating factors* instead of Corollary 1.14. An example is shown first.

Example 1.15 Consider the scalar first-order linear inhomogeneous equation

$$xy' + 2y = 4x^5.$$

Realize that after multiplying both sides by x, on the left-hand-side, a derivative of a product appears:

$$x^2y' + 2xy = 4x^6,$$

hence

$$(x^2y)' = 4x^6.$$

Integrating both sides we get

$$x^2y = \frac{4x^7}{7} + c,$$

hence

$$y = \frac{4x^5}{7} + \frac{c}{x^2}.$$

Note that the formula is a sum of two terms: $\frac{4x^5}{7}$, the particular solution of the inhomogeneous equation, and $\frac{c}{x^2}$, the general solution of the corresponding homogeneous equation. □

Consider the scalar first-order linear inhomogeneous equation

$$y' + r(x)y = f(x), \qquad x \in I \tag{1.5}$$

which corresponds to Eq. (1.3). We are looking for an integrating factor $\mu(x)$, so that if we multiply both sides of (1.5) by μ, a derivative of a product, more precisely, the derivative of $\mu(x)y$ appears. Now compare the left-hand-side of the equation

$$\mu(x)y' + \mu(x)r(x)y = \mu(x)f(x)$$

and the identity

$$(\mu(x)y)' = \mu(x)y' + \mu'(x)y.$$

Then the two expressions are the same if μ satisfies

$$\mu'(x) = \mu(x)r(x).$$

This is a linear homogeneous differential equation, so formula (1.4) yields that

$$\mu(x) = \mathrm{e}^{\int r(x)\,\mathrm{d}x} \tag{1.6}$$

is a possible solution. Hence multiplying (1.5) by μ we get

$$(\mu(x)y)' = \mu(x)f(x),$$

so

$$\mu(x)y = \int \mu(x)f(x)\,\mathrm{d}x,$$

and then
$$y = \frac{1}{\mu(x)} \int \mu(x) f(x)\,\mathrm{d}x.$$
Substituting the formula of μ and again writing $+c$ in the indefinite integral for the sake of an easier readability we get
$$y = \mathrm{e}^{-\int r(x)\,\mathrm{d}x} c + \mathrm{e}^{-\int r(x)\,\mathrm{d}x} \int \mathrm{e}^{\int r(x)\,\mathrm{d}x} f(x)\,\mathrm{d}x. \tag{1.7}$$

Let $x_0 \in I$. Instead of indefinite integral we use a particular antiderivative, the one which is given by a definite integral with lower limit x_0:
$$y = \mathrm{e}^{-\int_{x_0}^{x} r(t)\,\mathrm{d}t} c + \mathrm{e}^{-\int_{x_0}^{x} r(t)\,\mathrm{d}t} \int_{x_0}^{x} \mathrm{e}^{\int_{x_0}^{t} r(s)\,\mathrm{d}s} f(t)\,\mathrm{d}t.$$

For $x = x_0$ we get immediately that $y(x_0) = c$. Therefore, the solution of the linear inhomogeneous equation (1.5) corresponding to the IC
$$y(x_0) = y_0$$
is given by
$$\begin{aligned} y &= \mathrm{e}^{-\int_{x_0}^{x} r(t)\,\mathrm{d}t} y_0 + \mathrm{e}^{-\int_{x_0}^{x} r(t)\,\mathrm{d}t} \int_{x_0}^{x} \mathrm{e}^{\int_{x_0}^{t} r(s)\,\mathrm{d}s} f(t)\,\mathrm{d}t \\ &= \mathrm{e}^{-\int_{x_0}^{x} r(t)\,\mathrm{d}t} y_0 + \int_{x_0}^{x} \mathrm{e}^{\int_{x}^{t} r(s)\,\mathrm{d}s} f(t)\,\mathrm{d}t, \qquad x \in I. \end{aligned} \tag{1.8}$$

Formula (1.8) is called the *variation of constants formula*.

Example 1.16 Consider the IVP
$$x^2 y' - 4xy = 5x^3, \qquad y(1) = -2.$$
We apply the method of integrating factors. First we rewrite the equation into an explicit form, i.e., we divide it by x^2:
$$y' - \frac{4}{x} y = 5x. \tag{1.9}$$
We apply formula (1.6), so let
$$\mu(x) = \mathrm{e}^{\int -\frac{4}{x}\,\mathrm{d}x} = \mathrm{e}^{-4\ln x} = \mathrm{e}^{\ln x^{-4}} = x^{-4}.$$
Multiplying (1.9) by $\mu(x)$ we get
$$(x^{-4} y)' = 5x^{-3},$$
hence integrating both sides yields
$$x^{-4} y = \int 5x^{-3}\,\mathrm{d}x = -\frac{5}{2} x^{-2} + c,$$
so
$$y = -\frac{5}{2} x^2 + c x^4.$$
The IC implies
$$-2 = -\frac{5}{2} + c,$$
from which
$$c = \frac{1}{2}.$$

Therefore the solution of the IVP is

$$y = -\frac{5}{2}x^2 + \frac{1}{2}x^4.$$

□

1.4 Applications

In this section, we give some typical applications where first-order differential equations are used as models of a physical, engineering, biological, economical process. In all these applications, the unknown function is a function of the time, so it is natural to denote the independent variable by t. We will use this notation in this section.

Example 1.17 Experimental observations show that radioactive isotopes decay at a rate proportional to their mass. Suppose it is experienced that the mass of an isotope is reduced by 12% in 2 years. Give the mass of the isotope as the function of the time supposing that the initial mass is 10 mg. Find the time interval that is needed for the mass to decay to the half of its original size.

Let $Q = Q(t)$ denote the mass of the isotope at time t. We measure the time in years and the mass in mg. Then by the assumptions $Q(0) = 10$ and $Q(2) = 10 \cdot 0.88 = 8.8$. The assumption yields that the rate of change of the mass, i.e., the derivative of $Q(t)$ with respect to the time is proportional to the actual mass. Hence we get the equation

$$Q' = -kQ.$$

The derivative is negative, since the mass decreases, so the constant $k > 0$. By assumption the IC associated to the differential equation is

$$Q(0) = 10.$$

The equation is a first-order linear homogeneous differential equation, its general solution is

$$Q = c\mathrm{e}^{-kt}, \qquad t \geq 0.$$

Applying the IC we get

$$c = Q(0) = 10.$$

Therefore the solution of the IVP is

$$Q = 10\mathrm{e}^{-kt}, \qquad t \geq 0.$$

From the assumption $Q(2) = 8.8$ we get for $t = 2$

$$10\mathrm{e}^{-2k} = 8.8,$$

which yields

$$k = -\frac{\log 0.88}{2} \approx 0.027759,$$

and hence

$$Q = 10\mathrm{e}^{-kt} \approx 10\mathrm{e}^{-0.027759t}.$$

Let T denote the time interval which is needed that the original mass $Q(0)$ becomes its half, $\frac{1}{2}Q(0)$. Then

$$\frac{1}{2}Q(0) = Q(0)\mathrm{e}^{-kT}.$$

We can observe that the value of T does not depend on the initial size $Q(0)$ of the isotope, it depends only on k, i.e., on the decay rate of the isotope. Such a time is called the *half-life* of the material. We have

$$T = \frac{\ln 2}{k} \approx 10.844 \text{ years.}$$

□

Example 1.18 Suppose an initial investment S_0 is deposited in a bank account that pays interest at an annual rate r, i.e., $100r$ % at the end of the year ($0 < r < 1$). Then at the end of the year the value of the investment will be the sum of the initial investment and the interest payed:

$$S_0 + S_0 r = S_0(1 + r).$$

By the end of the second year, similar computation yields the balance

$$S_0(1 + r) + S_0(1 + r)r = S_0(1 + r)^2.$$

It is easy to see that after t years the balance of the account will be

$$S_0(1 + r)^t.$$

Next suppose that the bank pays the annual interest n times a year. Then at the end of the first term, i.e., after $\frac{1}{n}$ year the bank pays $\frac{1}{n}$ part of the annual interest, i.e., the amount $S_0\frac{r}{n}$. Therefore the balance of the account will be

$$S_0 + S_0\frac{r}{n} = S_0\left(1 + \frac{r}{n}\right).$$

After the 2nd period, i.e., after $\frac{2}{n}$ year the bank pays again the amount $S_0\left(1 + \frac{r}{n}\right)\frac{r}{n}$ as the interest, hence the balance will be

$$S_0\left(1 + \frac{r}{n}\right) + S_0\left(1 + \frac{r}{n}\right)\frac{r}{n} = S_0\left(1 + \frac{r}{n}\right)^2.$$

It is easy to see that after k periods the balance is

$$S_0\left(1 + \frac{r}{n}\right)^k,$$

and hence at the end of t years, i.e., after tn periods the balance is

$$S_0\left(1 + \frac{r}{n}\right)^{tn}.$$

It is known from calculus that

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{tn} = \mathrm{e}^{rt},$$

and the convergence is monotone increasing for all t.

Such a bank account where the balance at time t is given by the formula

$$S(t) = S_0\mathrm{e}^{rt}$$

is called a *continuously compounding interest* with annual rate r. Such a continuously compounding interest can be defined so that we assume that the rate of change of the balance $S(t)$ is proportional to the actual balance, i.e., equation

$$S' = rS$$

is satisfied. We can observe that this equation is identical to the equation of the radioactive decay, but the rate r is positive here, and the similar constant was negative in the case of the radioactive decay.

We now assume that we deposit the amount $S_0 = 20000$ EUR to an account which pays 4% annual interest computed continuously ($r = 0.04$), and we also withdraw $k = 100$ EUR from the account at a constant rate. Compute the balance of the account after 5 years.

S', the rate of change of the balance is the difference of the amount added (the interest payed) and withdrawn per unit time. Hence the equation

$$S' = rS - k \tag{1.1}$$

describes the balance of the account. This is a first-order linear inhomogeneous differential equation with constant coefficient, which can be solved with the method of integrating factors:

$$S' - rS = -k,$$

and so multiplying the equation by the factor $\mu(t) = \mathrm{e}^{-rt}$ we get

$$(\mathrm{e}^{-rt}S)' = -k\mathrm{e}^{-rt},$$

and hence

$$\mathrm{e}^{-rt}S = \frac{k}{r}\mathrm{e}^{-rt} + c.$$

Therefore the general solution is

$$S = c\mathrm{e}^{rt} + \frac{k}{r}.$$

The IC $S(0) = S_0$ yields $c = S_0 - \frac{k}{r}$, and so

$$S = \left(S_0 - \frac{k}{r}\right)\mathrm{e}^{rt} + \frac{k}{r}. \tag{1.2}$$

This formula shows that if the withdraw rate k is small, more precisely, if $k < rS_0$, then the coefficient of the exponential function is positive, so the function S is monotone increasing and it tends to ∞. In the critical case when the withdraw rate is $k = rS_0$, the balance is constant $S = \frac{k}{r}$ for $t \geq 0$. But if the withdrawal rate is larger than the critical value, i.e., $k > rS_0$, then the coefficient of the exponential function in (1.2) is negative, so S is monotone decreasing and it tends to 0. That implies that there exists a finite time T such that $S(T) = 0$, and, of course, no more withdraw is possible from the account.

Using the given parameters of this example we get $S(5) = 17500\mathrm{e}^{0.2} + 2500 = 23874.55$ EUR. □

Example 1.19 Suppose a mixing tank contains initially Q_0 kg of salt dissolved in 200 l of solution. A solution containing 0.2 kg/l salt flows into the tank with a constant rate of 4 l/min. We assume that the solution is well-stirred in the tank, and it flows out of the tank with a constant rate 4 l/min. Find the mass of the salt in the tank as a function of the time. Find the limiting value of the salt as $t \to \infty$.

The rate of change of the mass of the salt equals to the rate it enters minus the rate at which the salt leaves the tank. Let $Q = Q(t)$ denote mass of the salt in the tank at time t. The rate of change the salt enters to the tank is 0.2 kg/l $\cdot$ 4 l/min=0.8 kg/min. The rate of change the salt leaves the tank is $Q(t)/200$ kg/l $\cdot$ 4 l/min=Q(t)/50 kg/min. Therefore

$$Q' = 0.8 - \frac{1}{50}Q(t).$$

This is a first-order linear inhomogeneous differential equation whose solution is

$$Q = 40 + (Q_0 - 40)\mathrm{e}^{-\frac{1}{50}t}.$$

Therefore, $Q(t) \to 40$ as $t \to \infty$, so for large t the amount of salt in the tank is approximately 40 kg. □

Example 1.20 Suppose we drop a body of mass m from rest in a medium where a friction proportional to the speed applies to the body. For example we drop the body in a tank containing some fluid. Find the position and the velocity of the body as the function of the time.

We model the body as a material point and we apply Newton's Second Law $F = ma$, where F is the net force acting on the body and a is the acceleration of the body. We fix the vertical coordinate system so that the origin is at the position where we drop the body, and the positive direction is the downward direction. Let $v = v(t)$ denote the speed of the body, then $a = v'$. We consider two forces: the gravitational force which points downward, so it is positive, and the damping force which acts upward, so it is negative. Therefore the equation of the motion is a first-order linear inhomogeneous differential equation

$$mv' = mg - kv.$$

Its general solution is

$$v(t) = c\mathrm{e}^{-kt/m} + \frac{mg}{k}.$$

By the assumption, the motion starts form a rest, so the IC is $v(0) = 0$, which yields

$$v = \frac{mg}{k}\left(1 - \mathrm{e}^{-kt/m}\right).$$

Integrating the velocity, we get the displacement of the body, $x = x(t)$:

$$x = \int v(t)\,\mathrm{d}t = \int \frac{mg}{k}\left(1 - \mathrm{e}^{-kt/m}\right)\,\mathrm{d}t = \frac{mg}{k}t + \frac{m^2g}{k^2}\mathrm{e}^{-kt/m} + C.$$

The value of C can be determined from the condition $x(0) = 0$. A simple calculation gives $C = -\frac{m^2g}{k^2}$, and hence

$$x = \frac{mg}{k}t + \frac{m^2g}{k^2}\left(\mathrm{e}^{-kt/m} - 1\right)$$

is the displacement as the function of the time. Certainly, these formulas are valid until the body reaches the bottom of the tank, i.e., until $x(t) \le x_{\max}$, where $x_{\max}$ is the maximal displacement value. □

Example 1.21 Let $N = N(t)$ be the size of a biological population at time t. A population size can be the number of citizens of a country, the number of fish or animals in a territory, or the mass of a bacteria culture. It is common to describe the size of a population by a real function. For example, for a bacteria culture it gives the mass of the population, not the number of the bacteria.

Linear model: It is a common assumption that in a population both the *birth* and the *mortality rates*, i.e., the number of newborns and deads at unit time are proportional to the size of the population. Let r_b and r_m be the birth and mortality rates, respectively, and let $r := r_b - r_m$ be the *growth rate*. It is positive if the number of births is larger than the number of deads per unit time, and it is negative in the opposite case. Then the model equation is

$$N' = rN, \qquad N(0) = N_0. \tag{1.3}$$

Its solution is $N = N_0\mathrm{e}^{rt}$, which growths exponentially to $+\infty$ if the growth rate is positive, and it decays exponentially to 0 if the growth rate is negative. Eq. (1.3) was first introduced by *Malthus* as a model equation of a population in 1798. The rate of change divided by the number of population is called the *per capita growth rate*.

Suppose there is a continuous *migration* in the population with a constant rate M, where M = immigration rate − emigration rate. Then our model is

$$N' = rN + M, \qquad N(0) = N_0,$$

which is a linear inhomogeneous differential equation. This equation is identical to Eq. (1.1) (replacing k with $-M$), so its solution is

$$N = \left(N_0 + \frac{M}{r}\right) \mathrm{e}^{rt} - \frac{M}{r}.$$

Here the population dies out also in the case of a positive growth rate $r > 0$ if the migration rate is negative and it satisfies $M < -N_0 r$. If $M = -N_0 r$, then the size of the population remains constant.

Self-controlled nonlinear model: In real life, we cannot observe exponential growth of a population on a large time interval, since the limitations of food, space and other resources cannot allow the population to grow without any limitation. We get a more realistic model if we assume that if the size of the population is small, then the per capita growth rate is close to a constant, so the population growth exponentially. But for large population size the per capita growth rate should be negative since the large population size inhibits the growth rate. Moreover, it is natural to assume that the per capita growth rate is decreasing as the size of the population increasing. The most simple function satisfying the above assumptions is a linear one with a negative slope of the form $r - mN$, where $m > 0$ is a constant. This leads to the model

$$N' = rN - mN^2, \tag{1.4}$$

which is called *logistic differential equation*. This model was introduced by *Verhulst* in 1838. An other explanation for the particular form of the mortality part of the equation is that the number of individuals who die per unit time is proportional to N^2. The motivation for this is that the mortality rate reflects the number of encounters between individuals for searching for food and other resources. If the size of the population is N, then the number of encounters is $N(N-1) \approx N^2$.

Introduce the constant $K = r/m$. Then we have

$$N' = rN\left(1 - \frac{N}{K}\right), \tag{1.5}$$

which is the other standard form of the logistic differential equation. It is a separable differential equation, so we get

$$\int \frac{\mathrm{d}N}{N(1-N/K)} = \int r \, \mathrm{d}t.$$

Computing partial fractions in the first integral we get

$$\int \frac{\mathrm{d}N}{N(1-N/K)} = \int \left(\frac{1}{N} + \frac{1}{K}\frac{1}{1-\frac{N}{K}}\right) \mathrm{d}N = \ln|N| - \ln\left|1 - \frac{N}{K}\right| = \ln\left|\frac{N}{1-\frac{N}{K}}\right|.$$

Therefore

$$\ln\left|\frac{N}{1-\frac{N}{K}}\right| = rt + C,$$

which gives

$$\frac{N}{1-\frac{N}{K}} = c\mathrm{e}^{rt},$$

where $c = \pm \mathrm{e}^C$. The IC $N(0) = N_0$ yields

$$c = \frac{N_0}{1-\frac{N_0}{K}}.$$

Substituting back to the formula of the general solution gives

$$N = \frac{N_0 K}{N_0 + (K - N_0)\mathrm{e}^{-rt}}.$$

This shows that $N(t) \to K$ as $t \to \infty$. The constant K is called the *carrying capacity* of the environment, since if $0 < N_0 < K$, then $0 < N(t) < K$ holds for all $t > 0$. The integral curves of the logistic equation (1.4) with $r = 2$ and $m = \frac{1}{5}$, i.e., with carrying capacity $K = 10$ can be seen in Figure 1.1. The solutions starting from an initial condition below K approach to K monotone increasingly, and solutions starting above K approach to K monotone decreasingly. □

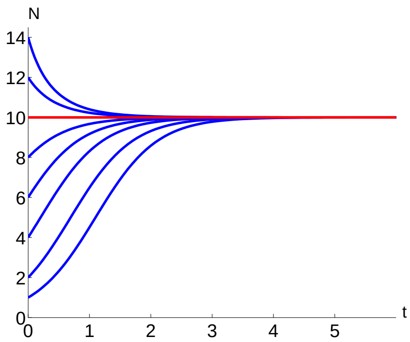

Figure 1.1: integral curves of Eq. (1.4), $r = 2$, $m = \frac{1}{5}$

1.5 First-order non-linear differential equations

A general form of an explicit first-order nonlinear differential equation is

$$y' = f(x, y). \tag{1.1}$$

If x and y are given, we can compute the right-hand-side of Eq. (1.1) at (x, y). This function value gives $y'(x)$, i.e., the slope of the tangent line to the graph of the solution. We can demonstrate it in the following way: take a grid of the plain, and at each grid point we evaluate the function value $f(x, y)$, and draw a line segment or a vector at this grid point with slope equal to $f(x, y)$. Such a figure is called *direction field*. A solution of Eq. (1.1) through a grid point is a curve whose tangent line is the line segment in the direction field. A direction field and a few solution curves of the logistic equation (1.5) with $r = 1$, $K = 1$ is given in Figure 1.2. Figure 1.3 shows a few solutions of the equation $y' = -y + x^2$ and its direction field.

Generating a direction field of a nonlinear equation is an easy task, and viewing at this figure, one can have a good guess about the flow of the solution curves. Hence it helps to conjecture the qualitative properties of the solution, e.g., the boundedness, convergence of solutions to a constant value, monotonicity properties. From Figure 1.2 we can observe that all solutions of the logistic equation $y' = y - y^2$ corresponding to initial values between 0 and 1 converge monotone increasingly to 1, and solutions starting from an initial condition larger than 1 converge to 1 monotone decreasingly.

We associate the IC

$$y(x_0) = y_0 \tag{1.2}$$

to Eq. (1.1), and suppose the function f is defined on a rectangular domain $[x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ for some $a > 0$ and $b > 0$.

The following basic theorem guarantees the existence of the solutions of our IVP (1.1)-(1.2).

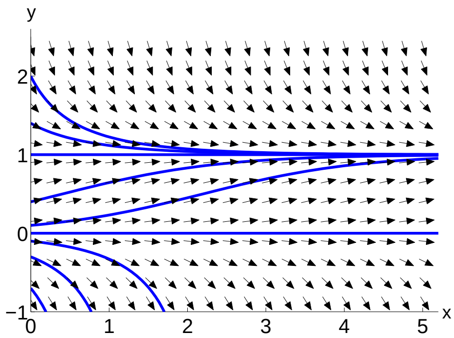

Figure 1.2: direction field of $y' = y - y^2$

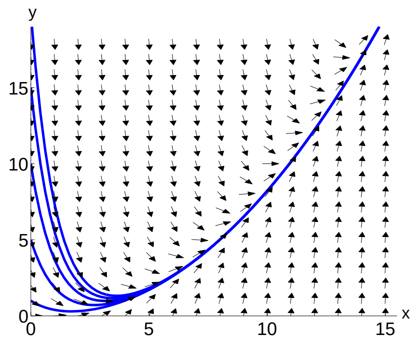

Figure 1.3: direction field of $y' = -y + x^2/10$

Theorem 1.22 (Peano) *Suppose $f\colon [x_0-a, x_0+a]\times[y_0-b, y_0+b] \to \mathbb{R}$ is continuous, and let M be its maximum, i.e., $M := \max\{|f(x,y)|\colon\ |x-x_0| \le a,\ |y-y_0| \le b\}$. Let $h := \min\left\{a, \frac{b}{M}\right\}$. Then the IVP (1.1)-(1.2) has at least one solution on the interval $I = [x_0-h, x_0+h]$.*

Example 1.23 Consider the IVP

$$y' = \sqrt{y}, \qquad y(0) = 0.$$

This is a separable ODE:

$$\frac{\mathrm{d}y}{\sqrt{y}} = \mathrm{d}x,$$

so integration yields

$$2\sqrt{y} = x + c.$$

Hence the general solution is

$$y = \frac{1}{4}(x+c)^2.$$

Using the IC we get $c = 0$, i.e.,

$$y = \frac{1}{4}x^2$$

is the solution of the IVP. On the other hand, we can see that $y = 0$ is also a solution of the IVP. Moreover, for all $C \ge 0$ the function

$$y(x) = \begin{cases} 0, & x \le C, \\ \frac{1}{4}(x-C)^2, & x > C \end{cases}$$

also satisfies the IVP. □

The above example shows that an IVP can have more than one solution, even infinitely many solutions. To guarantee uniqueness of the solution, we need a further assumption.

We say that a function $f\colon [x_0-a, x_0+a]\times[y_0-b, y_0+b] \to \mathbb{R}$ satisfies *Lipschitz property* or it is *Lipschitz continuous*, if there exists a constant $L \ge 0$ such that

$$|f(x,y) - f(x,\tilde{y})| \le L|y-\tilde{y}|, \qquad x \in [x_0-a, x_0+a], \quad y,\tilde{y} \in [y_0-b, y_0+b].$$

The next result shows that if in addition to the continuity, function f satisfies Lipschitz property, then the IVP has a unique solution.

Theorem 1.24 *Suppose the function* $f\colon [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] \to \mathbb{R}$ *is continuous and it satisfies the Lipschitz property. Let* $h := \min\left\{a, \frac{b}{M}\right\}$, *where* $M := \max\{|f(x, y)|\colon |x - x_0| \le a,\ |y - y_0| \le b\}$. *Then the IVP (1.1)-(1.2) has a unique solution on the interval* $I = [x_0 - h, x_0 + h]$.

1.6 System of first-order differential equations

In this section, we consider the system of first-order linear differential equations

$$\begin{aligned} y_1' &= f_1(x, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(x, y_1, \ldots, y_n) \end{aligned}$$

where $y_i = y_i(x)$ $(i = 1, \ldots, n)$ are the unknown functions. We associate the initial conditions

$$y_1(x_0) = z_1, \quad \ldots, \quad y_n(x_0) = z_n$$

to the system. Using the vector notations

$$\mathbf{y} = \mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix} \qquad \text{and} \qquad \mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} f_1(x, y_1, \ldots, y_n) \\ \vdots \\ f_n(x, y_1, \ldots, y_n) \end{pmatrix}$$

we can rewrite the system as

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \tag{1.1}$$

and the IC as

$$\mathbf{y}(x_0) = \mathbf{z}, \tag{1.2}$$

where $\mathbf{z} = (z_1, \ldots, z_n)^T$. Suppose $\mathbf{f}\colon U \to \mathbb{R}^n$, where $U \subset \mathbb{R} \times \mathbb{R}^n$ is an open set, and $(x_0, \mathbf{z}) \in U$.

Theorem 1.25 *Let* $U \subset \mathbb{R} \times \mathbb{R}^n$ *be an open set,* $\mathbf{f}\colon U \to \mathbb{R}^n$ *be continuous, and each components of* f *is continuously differentiable with respect to all arguments except for possibly the first variable. Then for all* $(x_0, \mathbf{z}) \in U$ *there exists* $h > 0$ *such that the IVP (1.1)-(1.2) has a unique solution on the interval* $[x_0 - h, x_0 + h]$.

Now we show that an nth-order scalar differential equation (1.2), and the corresponding IVP (1.2)-(1.3) is equivalent to a system of first-order linear differential equations (1.1)-(1.2). Introduce the variables

$$y_1(x) = y(x), \quad y_2(x) = y'(x), \quad y_3(x) = y''(x), \quad \ldots \quad y_n(x) = y^{(n-1)}(x).$$

Then clearly,

$$\begin{aligned} y_1'(x) &= y_2(x) \\ y_2'(x) &= y_3(x) \\ &\vdots \\ y_{n-1}'(x) &= y_n(x) \\ y_n'(x) &= f(x, y_1(x), y_2(x), \ldots, y_n(x)) \end{aligned}$$

hold. Define

$$\mathbf{y} = \mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_{n-1}(x) \\ y_n(x) \end{pmatrix}, \quad \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} \quad \text{and} \quad \mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ f(x, y_1, \dots, y_n). \end{pmatrix}.$$

Then $\mathbf{y}$ solves the IVP (1.1)-(1.2).

Let $I \subset \mathbb{R}$ be an open interval, and let $p_{n-1}, \dots, p_0 : I \to \mathbb{R}$ be continuous functions. Consider the nth-order scalar linear inhomogeneous differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x), \qquad x \in I \tag{1.3}$$

and the corresponding IC

$$y(x_0) = z_1, \quad y'(x_0) = z_2, \quad \dots, \quad y^{(n-1)}(x_0) = z_n, \tag{1.4}$$

where $z_1, z_2, \dots, z_n \in \mathbb{R}$. According to the previous transformation, we have

$$\mathbf{f}(x, \mathbf{y}) = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ -p_{n-1}(x)y_n - \cdots - p_1(x)y_2 - p_0(x)y_1 + f(x) \end{pmatrix}.$$

An application of Theorem 1.25 for the IVP (1.1)-(1.2) implies immediately the next result.

Theorem 1.26 *Let $p_{n-1}, \dots, p_0, f : I \to \mathbb{R}$ be continuous functions, $x_0 \in I$. The the IVP (1.3)-(1.4) has a unique solution on the interval I for all $z_1, \dots, z_n$.*

Chapter 2
Second-order differential equations

In this chapter, we discuss second-order scalar linear differential equations.

Let $I \subset \mathbb{R}$ be an open interval, $p, q, f\colon I \to \mathbb{R}$. The general form of an explicit *second-order linear inhomogeneous differential equation* is:

$$y'' + p(x)y' + q(x)y = f(x), \qquad x \in I. \tag{2.1}$$

The corresponding *second-order linear homogeneous differential equation* is

$$y'' + p(x)y' + q(x)y = 0, \qquad x \in I. \tag{2.2}$$

Given an initial time $x_0 \in I$ and initial values y_0 and y_0', we consider the IC

$$y(x_0) = y_0, \qquad y'(x_0) = y_0'. \tag{2.3}$$

The function f in Eq. (2.1) is sometimes called *forcing function*.

The following existence and uniqueness result follows from Theorem 1.24.

Theorem 2.1 *Let $p, q, f\colon I \to \mathbb{R}$ be continuous functions, $x_0 \in I$. Then the IVP (2.1)-(2.3) has a unique solution on I for every $y_0, y_0' \in \mathbb{R}$.*

2.1 Second-order linear homogeneous differential equations

In this section, we summarize the general properties of the second-order linear homogeneous differential equation (2.2). We can see that these properties are similar to those of the first-order linear homogeneous differential equations.

Theorem 2.2 *Let y_1 and y_2 be solutions of Eq. (2.2) on the interval I. Then the function $c_1y_1 + c_2y_2$ is also a solution of Eq. (2.2) on I for all $c_1, c_2 \in \mathbb{R}$ (or for $c_1, c_2 \in \mathbb{C}$), i.e., the set of solutions of the homogeneous equation (2.2) form a real (or complex) linear space.*

Proof: Substitute $y(x) = c_1y_1(x) + c_2y_2(x)$ to the left-hand-side of Eq. (2.2):

$$\begin{aligned}
&y''(x) + p(x)y'(x) + q(x)y(x) \\
&\qquad = (c_1y_1(x) + c_2y_2(x))'' + p(x)(c_1y_1(x) + c_2y_2(x))' + q(x)(c_1y_1(x) + c_2y_2(x)) \\
&\qquad = c_1y_1''(x) + c_2y_2''(x) + p(x)(c_1y_1'(x) + c_2y_2'(x)) + q(x)(c_1y_1(x) + c_2y_2(x)) \\
&\qquad = c_1\Big(y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)\Big) + c_2\Big(y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)\Big) \\
&\qquad = 0,
\end{aligned}$$

since y_1 and y_2 are both solutions of Eq. (2.2). $\square$

Let $y_1, y_2\colon I \to \mathbb{R}$ be given differentiable functions. The determinant

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the *Wronskian* of y_1 and y_2.

Let $y_1, y_2 : I \to \mathbb{R}$ be given functions. We say that the functions y_1 and y_2 are *linearly independent* if

$$c_1 y_1(x) + c_2 y_2(x) = 0, \qquad x \in I \tag{2.1}$$

holds if and only if $c_1 = c_2 = 0$. The functions y_1 and y_2 are *linearly dependent*, if they are not linearly independent.

We state the following two results without proof.

Theorem 2.3 *Let $y_1, y_2 : I \to \mathbb{R}$ be differentiable functions. Then y_1 and y_2 are linearly independent on the interval I if there exists $x_0 \in I$ such that $W(y_1, y_2)(x_0) \neq 0$. If y_1 and y_2 are linearly dependent on I, then $W(y_1, y_2)(x) = 0$ for all $x \in I$.*

Theorem 2.4 *Let y_1 and y_2 be solutions of Eq. (2.2) on I. Then either $W(y_1, y_2)(x) = 0$ for all $x \in I$ or $W(y_1, y_2)(x) \neq 0$ for all $x \in I$.*

The functions $y_1, y_2 : I \to \mathbb{R}$ are called *fundamental solutions* or *fundamental system* of Eq. (2.2) if y_1 and y_2 are solutions of Eq. (2.2) and y_1 and y_2 are linearly independent on I, i.e., $W(y_1, y_2)(x) \neq 0$, for $x \in I$.

Theorem 2.5 *Let y_1 and y_2 be fundamental solutions of Eq. (2.2) on I. Then for every initial values y_0 and y_0' there exist c_1 and c_2 such that $c_1 y_1 + c_2 y_2$ is the solution of the IVP (2.2)-(2.3).*

Proof: The linear system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= y_0' \end{aligned}$$

has a unique solution for c_1 and c_2, since y_1 and y_2 are fundamental solutions, so its Wronskian is not equal to 0 at x_0. Define the function $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for $x \in I$. Then y is also a solution of Eq. (2.2), and it satisfies the IC (2.3). Therefore, by Theorem 2.1, y is the unique solution of the IVP (2.2)-(2.3). □

Corollary 2.6 *The set of solutions of Eq. (2.2) is a two-dimensional linear space.*

2.2 Second-order linear homogeneous equations with constant coefficients

Consider the special case of Eq. (2.2) when all coefficients are constants. Let $a, b, c \in \mathbb{R}$, $a \neq 0$.

$$ay'' + by' + cy = 0, \qquad x \in \mathbb{R}. \tag{2.1}$$

Formula (1.4) shows that the solutions of a first-order linear homogeneous equation with constants coefficients are exponential functions. For our second-order equation (2.1), we also seek for exponential solutions of the form $y(x) = \mathrm{e}^{\lambda x}$, where λ is a real (or complex) constant. Then substituting $y'(x) = \lambda \mathrm{e}^{\lambda x}$ and $y''(x) = \lambda^2 \mathrm{e}^{\lambda x}$ into Eq. (2.1) we get

$$a\lambda^2 \mathrm{e}^{\lambda x} + b\lambda \mathrm{e}^{\lambda x} + c\mathrm{e}^{\lambda x} = 0,$$

which holds if and only if λ is a solution of the second-order algebraic equation

$$a\lambda^2 + b\lambda + c = 0. \tag{2.2}$$

Eq. (2.2) is called the *characteristic equation* of the differential equation (2.1).

We consider three cases:

Case 1: The characteristic equation (2.2) has two distinct real roots λ_1 and λ_2. Then the general solution of Eq. (2.1) is

$$y = c_1 \mathrm{e}^{\lambda_1 x} + c_2 \mathrm{e}^{\lambda_2 x}, \qquad x \in \mathbb{R}. \tag{2.3}$$

For this it is enough to check that the solutions $y_1 = \mathrm{e}^{\lambda_1 x}$ and $y_2 = \mathrm{e}^{\lambda_2 x}$ are linearly independent. Compute the Wronskian of y_1 and y_2:

$$W(y_1, y_2)(x) = \begin{vmatrix} \mathrm{e}^{\lambda_1 x} & \mathrm{e}^{\lambda_2 x} \\ \lambda_1 \mathrm{e}^{\lambda_1 x} & \lambda_2 \mathrm{e}^{\lambda_2 x} \end{vmatrix} = (\lambda_2 - \lambda_1)\mathrm{e}^{(\lambda_1+\lambda_2)x} \neq 0,$$

since $\lambda_1 \neq \lambda_2$, so y_1 and y_2 are linearly independent.

Case 2: The characteristic equation (2.2) has a double real root λ_0. This is satisfied if and only if $b^2 - 4ac = 0$, and then

$$\lambda_0 = -\frac{b}{2a}.$$

We show that beside of the solution $y_1 = \mathrm{e}^{\lambda_0 x}$, the function $y_2 = x\mathrm{e}^{\lambda_0 x}$ is also a solution of the differential equation. We have $y_2' = \mathrm{e}^{\lambda_0 x} + \lambda_0 x \mathrm{e}^{\lambda_0 x}$ and $y_2'' = 2\lambda_0 \mathrm{e}^{\lambda_0 x} + \lambda_0^2 x \mathrm{e}^{\lambda_0 x}$, so substituting into Eq. (2.1) we get

$$a(2\lambda_0 \mathrm{e}^{\lambda_0 x} + \lambda_0^2 x \mathrm{e}^{\lambda_0 x}) + b(\mathrm{e}^{\lambda_0 x} + \lambda_0 x \mathrm{e}^{\lambda_0 x}) + cx\mathrm{e}^{\lambda_0 x} = (a\lambda_0^2 + b\lambda_0 + c)x\mathrm{e}^{\lambda_0 x} + (2a\lambda_0 + b)\mathrm{e}^{\lambda_0 x} = 0,$$

which shows that y_2 is a solution of the homogeneous equation (2.1). On the other hand, y_1 and y_2 are linearly independent since their Wronskian is

$$W(y_1, y_2)(x) = \begin{vmatrix} \mathrm{e}^{\lambda_0 x} & x\mathrm{e}^{\lambda_0 x} \\ \lambda_0 \mathrm{e}^{\lambda_0 x} & \mathrm{e}^{\lambda_0 x} + \lambda_0 x \mathrm{e}^{\lambda_0 x} \end{vmatrix} = (1 + \lambda_0 x - \lambda_0 x)\mathrm{e}^{2\lambda_0 x} = \mathrm{e}^{2\lambda_0 x} \neq 0.$$

Therefore the general solution of Eq. (2.1) in this case is

$$y = c_1 \mathrm{e}^{\lambda_0 x} + c_2 x \mathrm{e}^{\lambda_0 x}, \qquad x \in \mathbb{R}. \tag{2.4}$$

Case 3: The characteristic equation (2.2) has two complex roots $\lambda_1 = \alpha + i\beta$ and its conjugate, $\lambda_2 = \alpha - i\beta$. Then, of course, $\tilde{y}_1(x) = \mathrm{e}^{\lambda_1 x}$ and $\tilde{y}_2(x) = \mathrm{e}^{\lambda_2 x}$ are solutions of Eq. (2.1), but they are complex valued functions:

$$\tilde{y}_1(x) = \mathrm{e}^{\lambda_1 x} = \mathrm{e}^{(\alpha+i\beta)x} = \mathrm{e}^{\alpha x}(\cos\beta x + i\sin\beta x),$$

and similarly,

$$\tilde{y}_2(x) = \mathrm{e}^{\lambda_2 x} = \mathrm{e}^{(\alpha-i\beta)x} = \mathrm{e}^{\alpha x}(\cos(-\beta x) + i\sin(-\beta x)) = \mathrm{e}^{\alpha x}(\cos\beta x - i\sin\beta x).$$

But then the functions

$$y_1(x) = \frac{1}{2}(\tilde{y}_1(x) + \tilde{y}_2(x)) = \mathrm{e}^{\alpha x}\cos\beta x$$

and

$$y_2(x) = \frac{1}{2i}(\tilde{y}_1(x) - \tilde{y}_2(x)) = \mathrm{e}^{\alpha x}\sin\beta x$$

are also solutions of Eq. (2.1), and they are real valued functions. We show that y_1 and y_2 are linearly independent:

$$\begin{aligned} W(y_1,y_2)(x) &= \begin{vmatrix} \mathrm{e}^{\alpha x}\cos\beta x & \mathrm{e}^{\alpha x}\sin\beta x \\ \alpha\mathrm{e}^{\alpha x}\cos\beta x - \beta\mathrm{e}^{\alpha x}\sin\beta x & \alpha\mathrm{e}^{\alpha x}\sin\beta x + \beta\mathrm{e}^{\alpha x}\cos\beta x \end{vmatrix} \\ &= \mathrm{e}^{2\alpha x}(\alpha\cos\beta x\sin\beta x + \beta\cos^2\beta x - \alpha\cos\beta x\sin\beta x + \beta\sin^2\beta x) \\ &= \mathrm{e}^{2\alpha x}\beta \\ &\neq 0, \end{aligned}$$

since $\beta \neq 0$. Therefore, the general solution of Eq. (2.1) in this case is

$$y = c_1\mathrm{e}^{\alpha x}\cos\beta x + c_2\mathrm{e}^{\alpha x}\sin\beta x, \qquad x \in \mathbb{R}. \tag{2.5}$$

Example 2.7 Solve the IVP

$$y'' + y' - 6y = 0, \qquad y(0) = -1, \quad y'(0) = 2.$$

The corresponding characteristic equation is

$$\lambda^2 + \lambda - 6 = 0,$$

which yields $\lambda_1 = -3$ and $\lambda_2 = 2$. Therefore the general solution of the ODE is

$$y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{2x}.$$

We need to use the IC to determine c_1 and c_2. First compute $y' = -3c_1\mathrm{e}^{-3x} + 2c_2\mathrm{e}^{2x}$. Substituting $x = 0$ to the formulas of the solution and its derivative, we get

$$\begin{array}{rcrcr} c_1 & + & c_2 & = & -1 \\ -3c_1 & + & 2c_2 & = & 2, \end{array}$$

whose solution is $c_1 = -4/5$ and $c_2 = -1/5$. Therefore the solution of the IVP is

$$y = -\frac{4}{5}\mathrm{e}^{-3x} - \frac{1}{5}\mathrm{e}^{2x}.$$

□

Example 2.8 Solve the IVP

$$4y'' + 12y' + 9y = 0, \qquad y(0) = -1, \quad y'(0) = 0.$$

The characteristic equation of this differential equation is

$$4\lambda^2 + 12\lambda + 9 = 0,$$

whose solution $\lambda_0 = -3/2$ is a double root. Then the general solution of the ODE is

$$y = c_1\mathrm{e}^{-\frac{3}{2}x} + c_2 x\mathrm{e}^{-\frac{3}{2}x}.$$

Compute $y' = -\frac{3}{2}c_1\mathrm{e}^{-\frac{3}{2}x} + c_2\mathrm{e}^{-\frac{3}{2}x} - \frac{3}{2}c_2 x\mathrm{e}^{-\frac{3}{2}x}$. Using the IC we get the linear system

$$\begin{array}{rcrcr} c_1 & & & = & -1 \\ -\frac{3}{2}c_1 & + & c_2 & = & 0 \end{array}$$

which yields $c_1 = -1$ and $c_2 = -\frac{3}{2}$, hence the solution of the IVP is

$$y = -\mathrm{e}^{-\frac{3}{2}x} - \frac{3}{2}x\mathrm{e}^{-\frac{3}{2}x}.$$

$\square$

Example 2.9 Consider the IVP

$$y'' - 2y' + 8y = 0, \qquad y(0) = 1, \quad y'(0) = -2.$$

The characteristic equation is

$$\lambda^2 - 2\lambda + 8 = 0,$$

which gives $\lambda = 1 \pm i\sqrt{7}$. Therefore the general solution of the ODE is

$$y = c_1\mathrm{e}^x \cos\sqrt{7}x + c_2\mathrm{e}^x \sin\sqrt{7}x.$$

First compute y':

$$y' = c_1\mathrm{e}^x \cos\sqrt{7}x - \sqrt{7}c_1\mathrm{e}^x \sin\sqrt{7}x + c_2\mathrm{e}^x \sin\sqrt{7}x + \sqrt{7}c_2\mathrm{e}^x \cos\sqrt{7}x.$$

The IC yields

$$\begin{array}{ccccc} c_1 & & & = & 1 \\ c_1 & + & \sqrt{7}c_2 & = & -2 \end{array}$$

and hence $c_1 = 1$ and $c_2 = -\frac{3}{\sqrt{7}}$. Therefore the solution of the IVP is

$$y = \mathrm{e}^x \cos\sqrt{7}x - \frac{3}{\sqrt{7}}\mathrm{e}^x \sin\sqrt{7}x.$$

$\square$

2.3 Second-order linear inhomogeneous differential equations

Consider again the linear inhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x), \qquad x \in I \tag{2.1}$$

and the associated linear homogeneous equation

$$y'' + p(x)y' + q(x)y = 0, \qquad x \in I. \tag{2.2}$$

Similarly to the first-order case it is easy to show the next results.

Theorem 2.10 *Let y_1 and y_2 be solutions of the inhomogeneous equation (2.1). Then the function $y = y_1 - y_2$ is a solution of the homogeneous equation (2.2).*

Theorem 2.11 *Let y_H be the general solution of the homogeneous equation (2.2), and y_{IP} be a particular solution of the inhomogeneous equation (2.1). Then the general solution of the inhomogeneous equation (2.1) is*

$$y_{IH} = y_H + y_{IP}.$$

The above theorem yields that we can compute the general solution of an inhomogeneous equation in two steps: first we compute the general solution of the homogeneous equation, and then it is enough to find a particular solution to the inhomogeneous equation. In the next two sections, we discuss two methods for finding particular solutions.

2.4 Method of undetermined coefficients

We first illustrate the *method of undetermined coefficients* on several examples and then we summarize the method.

Example 2.12 Find the general solution of the linear inhomogeneous differential equation

$$y'' - 2y' - 8y = 5\mathrm{e}^{-3x}. \tag{2.1}$$

According to Theorem 2.11, we have to find the general solution of the associated linear homogeneous equation and a particular solution to the linear inhomogeneous differential equation.

First we solve the linear homogeneous equation $y'' - 2y' - 8y = 0$. Its characteristic equation is $\lambda^2 - 2\lambda - 8 = 0$, which yields $\lambda_1 = -2$ and $\lambda_2 = 4$. Therefore the general solution of the homogeneous equation is

$$y_H = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{4x}.$$

Next we are looking for a particular solution. We need to find a function y that if we substitute to the left-hand-side of Eq. (2.1), then the linear combination of y'', y' and y with constant coefficients gives back the right-hand-side of the equation, i.e., the function $5\mathrm{e}^{-3x}$. It is natural to try to find a particular solution in the form

$$y_{IP} = A\mathrm{e}^{-3x}.$$

We call this function as a *test function*. Differentiating y_{IP} yields $y'_{IP} = -3A\mathrm{e}^{-3x}$ and $y''_{IP} = 9A\mathrm{e}^{-3x}$. Substituting these formulas into Eq. (2.1) we have

$$9A\mathrm{e}^{-3x} + 6A\mathrm{e}^{-3x} - 8A\mathrm{e}^{-3x} = 5\mathrm{e}^{-3x}.$$

Note that the above formula works since both the first and the second derivatives of the test function are exponential functions with exponent -3, so all terms after substitution into the equation are of the same type. Simplifying this equation we get $7A = 5$, hence $A = 5/7$. Therefore a possible particular solution of Eq. (2.1) is $y_{IP} = \frac{5}{7}\mathrm{e}^{-3x}$, and hence the general solution of the equation is

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{4x} + \frac{5}{7}\mathrm{e}^{-3x}.$$

Suppose the IC

$$y(0) = 5, \qquad y'(0) = 3$$

is also given. Then first compute $y' = -2c_1\mathrm{e}^{-2x} + 4c_2\mathrm{e}^{4x} - \frac{15}{7}\mathrm{e}^{-3x}$, and then using the general solution and the IC we get the algebraic system

$$\begin{aligned} c_1 + c_2 + \frac{5}{7} &= 5, \\ -2c_1 + 4c_2 - \frac{15}{7} &= 3. \end{aligned}$$

Solving the system we get $c_1 = 2$ and $c_2 = \frac{16}{7}$, hence the solution of the IVP is

$$y = 2\mathrm{e}^{-2x} + \frac{16}{7}\mathrm{e}^{4x} + \frac{5}{7}\mathrm{e}^{-3x}.$$

□

Example 2.13 Solve the linear inhomogeneous equation

$$y'' - 2y' - 8y = -25\cos 3x. \tag{2.2}$$

Since the left-hand-side of Eq. (2.2) is identical to that of Eq. (2.1), the general solution of the associated homogeneous equation is given in the previous example. We look for a particular solution in the form

$$y_{IP} = A\cos 3x + B\sin 3x.$$

Then $y'_{IP} = -3A\sin 3x + 3B\cos 3x$ and $y''_{IP} = -9A\cos 3x - 9B\sin 3x$. Plugging these formulas into Eq. (2.2) we get

$$-9A\cos 3x - 9B\sin 3x + 6A\sin 3x - 6B\cos 3x - 8A\cos 3x - 8B\sin 3x = -25\cos 3x,$$

and hence

$$(-17A - 6B)\cos 3x + (6A - 17B)\sin 3x = -25\cos 3x.$$

This will be an identity if and only if the coefficients of the same functions are identical on both sides of the equation, i.e.,

$$\begin{array}{rcrcr} -17A & - & 6B & = & -25 \\ 6A & - & 17B & = & 0. \end{array}$$

The solution of this linear system is $A = 17/13$ and $B = 6/13$, and hence the general solution of Eq. (2.2) is

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{4x} + \frac{17}{13}\cos 3x + \frac{6}{13}\sin 3x.$$

□

Example 2.14 Solve the equation

$$y'' - 2y' - 8y = 2x^2 - 3x + 1.$$

Now it is natural to look for a particular solution in the form

$$y_{IP} = Ax^2 + Bx + C.$$

Then $y'_{IP} = 2Ax + B$ and $y''_{IP} = 2A$. Substitution into the equation gives

$$2A - 4Ax - 2B - 8Ax^2 - 8Bx - 8C = 2x^2 - 3x + 1,$$

and hence

$$-8Ax^2 + (-4A - 8B)x + 2A - 2B - 8C = 2x^2 - 3x + 1.$$

Comparing the coefficients of the same functions on both sides we get

$$\begin{array}{rcrcrcr} -8A & & & & & = & 2 \\ -4A & - & 8B & & & = & -3 \\ 2A & - & 2B & - & 8C & = & 1, \end{array}$$

which yields $A = -1/4$, $B = 1/2$ and $C = -5/16$. Therefore the general solution of the equation is

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{4x} - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{5}{16}.$$

□

Example 2.15 Solve the equation

$$y'' - 2y' - 8y = 9\mathrm{e}^{-2x}\sin 3x.$$

Here we look for a particular solution in the form

$$y_{IP} = A\mathrm{e}^{-2x}\cos 3x + B\mathrm{e}^{-2x}\sin 3x.$$

Then

$$\begin{aligned} y'_{IP} &= -2A\mathrm{e}^{-2x}\cos 3x - 3A\mathrm{e}^{-2x}\sin 3x - 2B\mathrm{e}^{-2x}\sin 3x + 3B\mathrm{e}^{-2x}\cos 3x \\ &= (-2A + 3B)\mathrm{e}^{-2x}\cos 3x + (-3A - 2B)\mathrm{e}^{-2x}\sin 3x, \end{aligned}$$

and

$$\begin{aligned}
y''_{IP} &= (4A-6B)\mathrm{e}^{-2x}\cos 3x + (6A-9B)\mathrm{e}^{-2x}\sin 3x + (6A+4B)\mathrm{e}^{-2x}\sin 3x \\
&\quad +(-9A-6B)\mathrm{e}^{-2x}\cos 3x \\
&= (-5A-12B)\mathrm{e}^{-2x}\cos 3x + (12A-5B)\mathrm{e}^{-2x}\sin 3x.
\end{aligned}$$

Therefore substitution into the equation and comparing the coefficients of the same functions on both sides of the equation give after a sort calculation

$$\begin{array}{rcrcl}
18A & - & 9B & = & 0 \\
-9A & - & 18B & = & 9.
\end{array}$$

The solution of the linear system is $A = 2/5$ and $B = -1/5$, and so the general solution of the differential equation is

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{4x} + \frac{2}{5}\mathrm{e}^{-2x}\cos 3x - \frac{1}{5}\mathrm{e}^{-2x}\sin 3x.$$

□

Example 2.16 Consider the linear inhomogeneous ODE

$$y'' - 2y' - 8y = 7\mathrm{e}^{4x}. \tag{2.3}$$

We look for a particular solution in the form $y_{IP} = A\mathrm{e}^{4x}$, as we did in Example 2.12. Then plugging the derivatives $y'_{IP} = 4A\mathrm{e}^{4x}$ and $y''_{IP} = 16A\mathrm{e}^{4x}$ into Eq. (2.3) yields

$$16A\mathrm{e}^{4x} - 8A\mathrm{e}^{4x} - 8A\mathrm{e}^{4x} = 7\mathrm{e}^{4x},$$

which is a contradiction. Therefore Eq. (2.3) has no solution in the above form. We could have realized it without the above computation since e^{4x} solves the homogeneous equation, therefore it cannot be the solution of any corresponding inhomogeneous equation.

Therefore we have to modify our test function. Let us try the function $y_{IP} = Ax\mathrm{e}^{4x}$, which is also similar to the right-hand-side. Then $y'_{IP} = A\mathrm{e}^{4x} + 4Ax\mathrm{e}^{4x}$ and $y''_{IP} = 8A\mathrm{e}^{4x} + 16Ax\mathrm{e}^{4x}$, therefore after substitution to the left-hand-side of Eq. (2.3) the function e^{4x} appears, so there is a chance that the two sides of the equation can be equal:

$$8A\mathrm{e}^{4x} + 16Ax\mathrm{e}^{4x} - 2A\mathrm{e}^{4x} - 8Ax\mathrm{e}^{4x} - 8Ax\mathrm{e}^{4x} = 7\mathrm{e}^{4x}.$$

After simplification we get

$$6A\mathrm{e}^{4x} = 7\mathrm{e}^{4x},$$

and so $A = 7/6$. Therefore the general solution of the Eq. (2.3) is

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{4x} + \frac{7}{6}x\mathrm{e}^{4x}.$$

□

The above examples demonstrate that in the cases when the right-hand-side of

$$ay'' + by' + cy = f(x), \qquad x \in \mathbb{R}$$

is exponential, sine, cosine or polynomial functions, then the test function y_{IP} in the method of undetermined coefficients can be selected according to Table 2.1.

Here $A, B, A_n, \ldots, A_0$ are unknown constants to be determined, and the test function should be given by the formula using exponent $s = 0$. If that formula does not work (since it is a solution of the corresponding homogeneous equation) then we multiply it by x, i.e., use $s = 1$ in

Table 2.1: Test functions

$f(x)$	y_{IP}
$a\mathrm{e}^{\alpha x}$	$A\mathrm{e}^{\alpha x}x^s$ $(s = 0, 1, 2)$
$a\cos\beta x + b\sin\beta x$	$(A\cos\beta x + B\sin\beta x)x^s$ $(s = 0, 1, 2)$
$a_n x^n + \cdots + a_1 x + a_0$	$(A_n x^n + \cdots + A_1 x + A_0)x^s$ $(s = 0, 1, 2)$

the table. In case if it still does not work, we use exponent $s = 2$. It can be proved that one of the above test functions will always give a particular solution, i.e., there can be find constants which yield a solution.

Generalizing Example 2.15 one can prove that the method of undetermined coefficients also works in the case when the forcing function is a product of two or three functions from the classes of exponential, sine, cosine or polynomial functions. Then the test function is a product of the test functions from Table 2.1 (of course, omitting superfluous constants).

Theorem 2.17 (principle of superposition) *Let y_1 and y_2 be solutions of*

$$y_1'' + p(x)y_1' + q(x)y_1 = f_1(x), \qquad x \in I,$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 = f_2(x), \qquad x \in I,$$

respectively. Then the function $y(x) = y_1(x) + y_2(x)$ is a solution of equation

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x), \qquad x \in I.$$

Proof: Plugging $y = y_1 + y_2$ into the equation gives:

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (y_1 + y_2)'' + p(x)(y_1 + y_2)' + q(x)(y_1 + y_2) \\ &= y_1'' + p(x)y_1' + q(x)y_1 + y_2'' + p(x)y_2' + q(x)y_2 \\ &= f_1(x) + f_2(x). \end{aligned}$$

□

Example 2.18 Solve the linear ODE

$$y'' - 2y' - 8y = 5\mathrm{e}^{-3x} - 25\cos 3x + 2x^2 - 3x + 1.$$

Note that in earlier examples, we solved the inhomogeneous equations with right-hand-side equal to $5\mathrm{e}^{-3x}$, $-25\cos 3x$ and $2x^2 - 3x + 1$, respectively. Therefore the particular solutions obtained earlier and Theorem 2.17 yield the general solution

$$y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{4x} + \frac{5}{7}\mathrm{e}^{-3x} + \frac{17}{13}\cos 3x + \frac{6}{13}\sin 3x - \frac{1}{4}x^2 + \frac{1}{2}x - \frac{5}{16}.$$

□

2.5 Method of variation of parameters

Consider the inhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x), \qquad x \in I. \tag{2.1}$$

Suppose we know the fundamental solutions y_1 and y_2 of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0, \qquad x \in I, \tag{2.2}$$

i.e., the general solution

$$y_H = c_1 y_1(x) + c_2 y_2(x)$$

of the homogeneous equation is known. The idea of our method is to try to find a particular solution of the inhomogeneous equation in the form

$$y_{IP} = u_1(x)y_1(x) + u_2(x)y_2(x).$$

We are looking for coefficient functions u_1 and u_2 which yield a particular solution. This method is called the method of *variation of parameters* or *variation of constants.* Then we have

$$y'_{IP} = u'_1(x)y_1(x) + u_1(x)y'_1(x) + u'_2(x)y_2(x) + u_2(x)y'_2(x).$$

If we differentiate this expression once more and plug it into Eq. (2.1) then we get one equation with two unknowns. So this calculation is undetermined. But this means we can specify an additional constrain on the coefficients. The key idea is that we suppose u_1 and u_2 satisfy relation

$$u'_1(x)y_1(x) + u'_2(x)y_2(x) = 0, \qquad x \in I. \tag{2.3}$$

This assumption simplifies the calculation significantly since then

$$y'_{IP} = u_1(x)y'_1(x) + u_2(x)y'_2(x),$$

and hence

$$y''_{IP} = u'_1(x)y'_1(x) + u_1(x)y''_1(x) + u'_2(x)y'_2(x) + u_2(x)y''_2(x).$$

Therefore substituting into Eq. (2.1) we have

$$\begin{aligned}
&u'_1(x)y'_1(x) + u_1(x)y''_1(x) + u'_2(x)y'_2(x) + u_2(x)y''_2(x)\\
&\qquad + p(x)(u_1(x)y'_1(x) + u_2(x)y'_2(x)) + q(x)(u_1(x)y_1(x) + u_2(x)y_2(x))\\
&= u_1(x)(y''_1(x) + p(x)y'_1(x) + q(x)y_1(x)) + u_2(x)(y''_2(x) + p(x)y'_2(x) + q(x)y_2(x))\\
&\qquad + u'_1(x)y'_1(x) + u'_2(x)y'_2(x)\\
&= u'_1(x)y'_1(x) + u'_2(x)y'_2(x).
\end{aligned}$$

Therefore, together with Eq. (2.3), u_1 and u_2 satisfy the system

$$u'_1(x)y_1(x) + u'_2(x)y_2(x) = 0 \tag{2.4}$$
$$u'_1(x)y'_1(x) + u'_2(x)y'_2(x) = f(x). \tag{2.5}$$

For u'_1 and u'_2 it is a system of linear equations. It has a unique solution, since the determinant of the coefficients is the Wronskian of y_1 and y_2, and $W(y_1, y_2)(x) \neq 0$, since y_1 and y_2 are linearly independent. Then integration gives the formulas of u_1 and u_2.

Example 2.19 It can be checked easily that the general solution of the homogeneous equation

$$x^2y'' + 2xy' - 6y = 0, \qquad x > 0$$

is

$$y_H = \frac{c_1}{x^3} + c_2 x^2.$$

We solve the inhomogeneous equation

$$x^2 y'' + 2xy' - 6y = x^3, \qquad x > 0$$

using the method of variation of parameters. We look for a particular solution in the form

$$y_{IP} = \frac{u_1}{x^3} + u_2 x^2$$

where u_1 and u_2 are unknown functions (we omit the arguments of u_1 and u_2 for simplicity). First we divide the equation by x^2 to rewrite it in the explicit form (2.1):

$$y'' + \frac{2}{x} y' - \frac{6}{x^2} y = x, \qquad x > 0.$$

Then using the system (2.4)-(2.5) we get

$$\begin{aligned} \frac{u_1'}{x^3} + u_2' x^2 &= 0 \\ -3\frac{u_1'}{x^4} + 2u_2' x &= x. \end{aligned}$$

Computing the solutions of the algebraic system we get $u_1' = -\frac{1}{5}x^5$ and $u_2' = \frac{1}{5}$, so $u_1 = \int -\frac{1}{5}x^5 \, \mathrm{d}x = -\frac{1}{30}x^6$ and $u_2 = \int \frac{1}{5} \, \mathrm{d}x = \frac{1}{5}x$. We note that here the constants of the integrations were omitted since we needed only one possible u_1 and u_2, not all solutions. Therefore

$$y_{IP} = -\frac{1}{30}x^6 \frac{1}{x^3} + \frac{1}{5}x \cdot x^2 = \frac{1}{6}x^3,$$

so the general solution is

$$y = \frac{c_1}{x^3} + c_2 x^2 + \frac{1}{6}x^3, \qquad x > 0.$$

□

2.6 Applications

In this section, we discuss two classical systems of mechanics, a spring-mass system and the pendulum. We show that the mathematical models of the motion of the above mechanical systems can be given by second-order differential equations.

Example 2.20 (spring-mass system) Consider a vertical spring of length l (see Figure 2.1). We attach a body of mass m to the spring at its centroid, which causes an elongation L.

Without applying any external disturbance the mass is at rest, it hangs in the same position, which we call *static equilibrium*. In this case two forces act on the body at its centroid: the gravitational force of magnitude mg which points downward, and the force due to the spring acting upward. Hooke's law yields that the spring force is proportional to the elongation. We fix a vertical coordinate system so that the downward direction is the positive direction and the origin is at the position of the static equilibrium. Then the spring force at the static equilibrium is negative and it equals to $F_s = -kL$, where k is the spring constant. At the equilibrium, the two forces balance each other, so $mg - kL = 0$.

If the body is pulled forward down or pushed up, and then it is released, or a force is applied to the body continuously, it will be in motion. Our goal is to describe the displacement of the body measured from the equilibrium position as a function of the time. Let $x(t)$ denote the

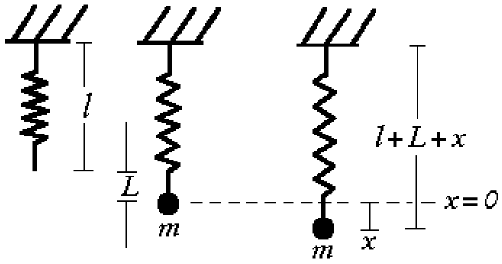

Figure 2.1: spring-mass system

displacement of the body: it is positive if the spring is extended, and it is negative, if the spring is compressed. We apply Newton's law of motion

$$F = ma,$$

where F is the net force acting on the body, and a is the acceleration of the body, both are functions of time. We have $a = v' = x''$, where $v = x'$ is the speed of the body. We consider the following forces on the body:

(i) the gravitational force always acts downward, so in the positive direction, its value is mg.

(ii) the spring force acts to restore the spring to its original position, i.e., if the spring is expanded, it acts upwards, so in the negative direction, and if the spring is compressed, it acts downward, so in the positive direction. In both cases $F_s = -k(L + x)$ gives the correct sign of the force, $L + x$ is the total elongation of the spring.

(iii) We assume that the motion of the spring-mass system is located in a fluid where damping cannot be neglected. Then a damping force is applied to the body which always act opposite to the direction of the velocity of the body. Experiment gives that (in most circumstances) magnitude of the damping force is proportional to the velocity of the body. Then $F_d = -\gamma v = -\gamma x'$ is the formula of the damping force, where γ is the damping constant. If the speed is positive, then the body moves downward, and the damping force acts upward. In the case when the speed is negative, then the body moves upward, and the damping force acts downward, so it is positive.

(iv) There can be any external force $f(t)$ acting on the body, which can be even time-dependent.

Then $F(t) = mg - k(L + x) - \gamma x' + f(t)$. Using Newton's law and relation $mg = kL$ we get the second-order linear differential equation

$$mx'' + \gamma x' + kx = f(t), \qquad t \geq 0. \tag{2.1}$$

This is a second-order linear differential equation with constant coefficients in the homogeneous part. We associate the IC

$$x(0) = x_0, \qquad x'(0) = v_0 \tag{2.2}$$

to the ODE. Here x_0 is the initial displacement, and v_0 is the initial speed of the body. These are values which can be measured, so it is reasonable to assume that they are known quantities. □

Example 2.21 (harmonic oscillation) In this example, we consider a special case of Eq. (2.1): we assume that damping can be omitted, i.e., $\gamma = 0$, and there is no external force, i.e., $f(t) = 0$. Then the equation simplifies to the linear homogeneous equation

$$mx'' + kx = 0. \tag{2.3}$$

Its characteristic equation is

$$m\lambda^2 + k = 0,$$

whose solutions are $\lambda = \pm i\omega_0$, where $\omega_0 = \sqrt{\frac{k}{m}}$. Hence the general solution of Eq. (2.3) is

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

This yields a harmonic oscillation for any c_1 and c_2, since simple manipulations imply

$$x(t) = R \cos(\omega_0 t - \delta).$$

We have

$$R\cos(\omega_0 t - \delta) = R(\cos \omega_0 t \cos \delta + \sin \omega_0 t \sin \delta) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

if

$$R \cos \delta = c_1 \qquad \text{and} \qquad R \sin \delta = c_2,$$

i.e.,

$$R = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \operatorname{tg} \delta = \frac{c_2}{c_1}.$$

ω_0 is called *natural frequency* or *natural angular frequency*, δ is called *phase* or *phase angle*, and R is called the *amplitude* of the oscillation.

□

Example 2.22 (damped oscillation) Here we assume that damping cannot be neglected, but no external force acts on the body. So we assume $\gamma > 0$, and consider

$$mx'' + \gamma x' + kx = 0. \tag{2.4}$$

The corresponding characteristic equation is

$$m\lambda^2 + \gamma\lambda + k = 0.$$

We discuss three cases. (i) $\gamma^2 - 4mk > 0$ (large damping). Then

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} < 0 \qquad \text{and} \qquad \lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} < 0$$

are the two real characteristic roots, and therefore the general solution of Eq. (2.4) is

$$x(t) = c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t}.$$

From the fact that both λ_1 and λ_2 are negative it follows that $x(t) \to 0$ as $t \to \infty$. In this case the body tends to its equilibrium position with an exponential speed. Figure 2.2 shows a few typical solutions of the equation. The solutions are generated form the initial conditions $x(0) = 1, x'(0) = 1$; $x(0) = 0.5, x'(0) = -0.5$; $x(0) = -0.5, x'(0) = 1$ and $x(0) = -1, x'(0) = 0$ in all the three figures.

(ii) $\gamma^2 - 4mk = 0$ (critical damping). Here $\lambda = -\frac{\gamma}{2m}t$ is the real and double characteristic root, therefore the general solution is

$$x(t) = c_1 \mathrm{e}^{-\frac{\gamma}{2m}t} + c_2 t \mathrm{e}^{-\frac{\gamma}{2m}t}.$$

In this case the motion also tends to the equilibrium position. Figure 2.3 displays some typical solution curves.

(iii) $\gamma^2 - 4mk < 0$ (small damping). In this case, there are two complex roots of the characteristic equation:

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm i\mu, \qquad \text{where} \quad \mu = \frac{\sqrt{4km - \gamma^2}}{2m},$$

and hence the general solution is

$$x(t) = \mathrm{e}^{-\frac{\gamma}{2m}t}\Big(c_1 \cos \mu t + c_2 \sin \mu t\Big).$$

The solutions tend to 0, but now the motion is oscillatory, see Figure 2.4. □

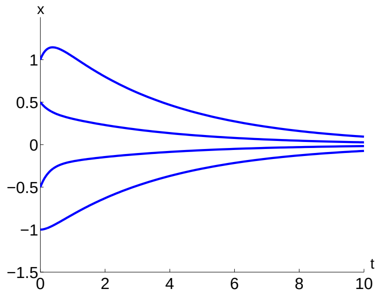

Figure 2.2: $x'' + 4x' + x = 0$

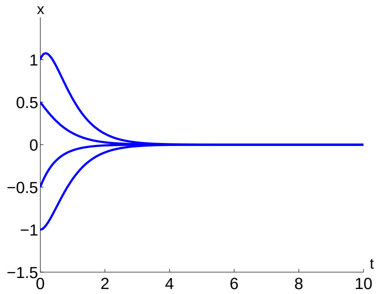

Figure 2.3: $x'' + 4x' + 4x = 0$

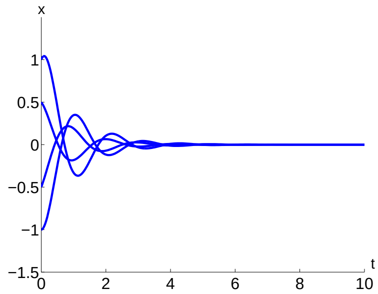

Figure 2.4: $x'' + 2x' + 10x = 0$

Example 2.23 (amplitude modulation) Now consider the case when there is no damping, but a periodic external force is applied to the body. More specifically, consider the equation

$$mx'' + kx = a\cos\omega t. \tag{2.5}$$

Let ω_0 be the *natural frequency* of the system, i.e., $\omega_0 = \sqrt{\frac{k}{m}}$. First, we suppose $\omega \neq \omega_0$. We use the method of undetermined coefficients to find a particular solution of Eq. (2.5) in the form

$$x_{IP} = A\cos\omega t + B\sin\omega t.$$

Then $x'_{IP} = -A\omega\sin\omega t + B\omega\cos\omega t$ and $x''_{IP} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$. Therefore substituting the above expressions into Eq. (2.5) we get

$$(-mA\omega^2 + kA)\cos\omega t + (-mB\omega^2 + kB)\sin\omega t = a\cos\omega t,$$

hence

$$A = \frac{a}{k - m\omega^2} = \frac{a}{(\omega_0^2 - \omega^2)m} \qquad \text{and} \qquad B = 0.$$

Therefore the general solution of Eq. (2.5) is

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{a}{(\omega_0^2 - \omega^2)m}\cos\omega t.$$

Suppose that we start the motion of the spring-mass system from rest, i.e., from the IC $x(0) = 0$ and $x'(0) = 0$. Short calculation yields $c_1 = -\frac{a}{(\omega_0^2-\omega^2)m}$ and $c_2 = 0$, so the solution of the IVP is

$$x(t) = \frac{a}{(\omega_0^2 - \omega^2)m}\Big(\cos\omega t - \cos\omega_0 t\Big).$$

Simple trigonometric manipulations imply

$$x(t) = \frac{2a}{(\omega_0^2 - \omega^2)m}\sin\frac{(\omega_0 - \omega)t}{2}\sin\frac{(\omega_0 + \omega)t}{2}.$$

If $\omega \approx \omega_0$, then $\sin\frac{(\omega_0+\omega)t}{2}$ oscillates much faster than the function $\sin\frac{(\omega_0-\omega)t}{2}$, so we can see two oscillations in the graph of the solution. The expression $\frac{2a}{(\omega_0^2-\omega^2)m}\sin\frac{(\omega_0-\omega)t}{2}$ can be considered as a slowly varying amplitude of the fast oscillation $\sin\frac{(\omega_0+\omega)t}{2}$, see Figure 2.5. This phenomenon in electronics is called *amplitude modulation*, and such a curve is called a *beat*. □

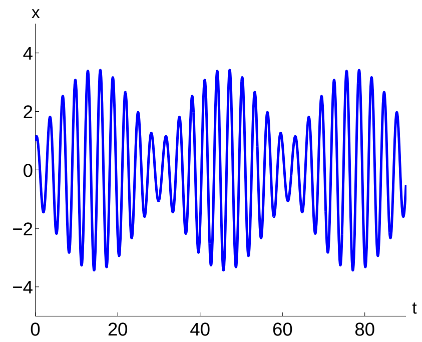

Figure 2.5: $x'' + 4x = \cos 2.2t,\ x(0) = 1 = x'(0)$

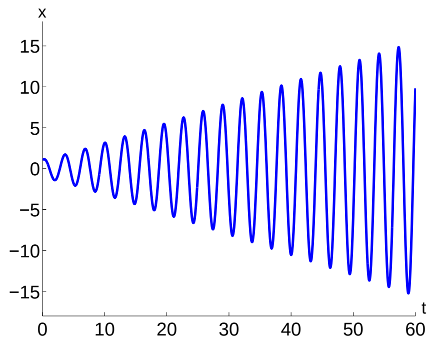

Figure 2.6: $x'' + 4x = \cos 2t,\ x(0) = 1 = x'(0)$

Example 2.24 (resonance) Now we consider again the above example in the case when the forcing frequency is equal to the natural frequency of the system, i.e., $\omega = \omega_0$. In this case the test function of the previous example is a solution of the homogeneous equation. Therefore here we are looking for a particular solution in the form

$$x_{IP} = t(A\cos\omega_0 t + B\sin\omega_0 t).$$

A little calculation gives that the function

$$x = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{a}{2m\omega_0} t\sin\omega_0 t$$

is the general solution of Eq. (2.5). This is an oscillatory function, but it is not bounded, see Figure 2.6. Such a phenomenon is called *resonance*. In practice, of course, the spring breaks if the elongation becomes large. Also, the assumption that the spring force depends linearly on the displacement is valid only for small elongation only. □

Example 2.25 (forced vibration with damping) Suppose that the damping can not be neglected and a periodic forcing function acts to the body:

$$mx'' + \gamma x' + kx = a\cos\omega t. \tag{2.6}$$

We look for a particular solution in the form

$$x_{IP} = A\cos\omega t + B\sin\omega t.$$

Computing $x'_{IP} = -A\omega\sin\omega t + B\omega\cos\omega t$ and $x''_{IP} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$ and substituting them to Eq. (2.6) we get

$$(-mA\omega^2 + \gamma B\omega + kA)\cos\omega t + (-mB\omega^2 - \gamma A\omega + kB)\sin\omega t = a\cos\omega t,$$

hence

$$\begin{aligned} -mA\omega^2 + \gamma B\omega + kA &= a \\ -mB\omega^2 - \gamma A\omega + kB &= 0. \end{aligned}$$

It follows from the definition of ω_0 that $k = m\omega_0^2$, therefore we have

$$\begin{aligned} mA(\omega_0^2 - \omega^2) + \gamma B\omega &= a \\ mB(\omega_0^2 - \omega^2) - \gamma A\omega &= 0. \end{aligned}$$

Its solution is

$$A = \frac{am(\omega_0^2 - \omega^2)}{m(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad \text{and} \qquad B = \frac{a\gamma\omega}{m(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2},$$

so the particular solution is

$$x_{IP} = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \delta),$$

where

$$R = \sqrt{A^2 + B^2} = \frac{|a|}{\sqrt{m(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \qquad \text{and} \qquad \operatorname{tg}\delta = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}.$$

Suppose, e.g., that Case (iii) of Example 2.22 holds, i.e., the damping is small. Using the notation of Example 2.25 we get that the general solution is

$$x(t) = x_H(t) + x_{IP}(t) = \mathrm{e}^{-\frac{\gamma}{2m}t}(c_1\cos\mu t + c_2\sin\mu t) + R\cos(\omega t - \delta).$$

Here the solution can be viewed as a sum of two functions. We have seen in Example 2.25 that the solution of the homogeneous equation, $x_H(t)$, tends to 0 as $t \to \infty$ in all the three cases. This part of the solution is called *transient solution*. Therefore, for large t, this part of the solution is negligible, and $x(t) \approx x_{IP}(t)$. Hence starting from any IC, for large t the solution becomes close to a periodic function with frequency equals to the frequency of the forcing function. We say that the solution tends to a *periodic steady state* or to a *forced response*. □

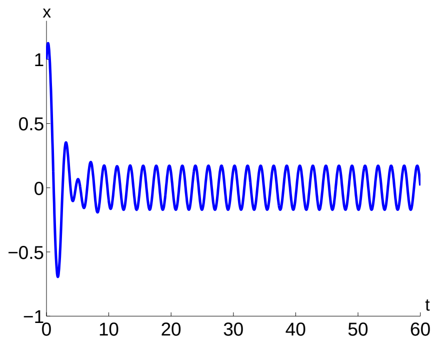

Figure 2.7: $x'' + x' + 4x = \cos 3t,\ x(0) = 1 = x'(0)$

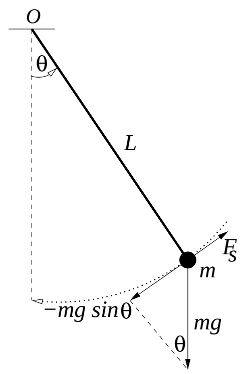

Figure 2.8: mathematical pendulum

Example 2.26 (the pendulum) Consider a mathematical pendulum, i.e., a pendulum where the mass m is suspended at the end of a weightless rod of length L, and the rod is attached to a frictionless pivot, see Figure 2.8. Then the mass moves along a circle of radius L. The angle of the rod from the vertical direction measured in radians is denoted by $\theta = \theta(t)$. We assume the positive direction is the counterclockwise direction. Therefore, if the rod is rotated by angle θ then the distance done by the body along the circumference of the circle of radius L is $s = L\theta$. The circumferential speed of the body is $v = L\theta'$, and its circumferential acceleration is $a = L\theta''$, both are tangential to the circle. We apply Newton's Second Law computing the forces in the tangential direction.

We consider three forces on the body, the gravitational force mg, the force F_r acting by the rod to the body, and the damping force F_d. The gravitational force acts in the downward vertical direction, its tangential component is $-mg\sin\theta$, see Figure 2.8. We suppose that the damping force is proportional to the velocity, and its direction is opposite to the direction of the motion. So we have $F_s = -\gamma L\theta'$. We assume no other force acts to the body. Then Newton's Second Law yields

$$mL\theta'' = -\gamma L\theta' - mg\sin\theta,$$

which gives

$$\theta'' + \frac{\gamma}{m}\theta' + \frac{g}{L}\sin\theta = 0, \qquad t \geq 0. \tag{2.7}$$

This is a second-order nonlinear differential equation since $\sin\theta$ appears in the equation. We associate the IC

$$\theta(0) = \theta_0 \qquad \text{and} \qquad \theta'(0) = \theta'_0$$

to Eq. (2.7), which is the initial angle θ_0 and the initial angular velocity θ'_0.

It is known that if $\theta \approx 0$ then $\sin\theta \approx \theta$. So for small angles Eq. (2.7) can be approximated by the linear equation

$$\theta'' + \frac{\gamma}{m}\theta' + \frac{g}{L}\theta = 0, \qquad t \geq 0. \tag{2.8}$$

This is a second-order linear homogeneous equation with positive constant coefficients, and this equation is identical to the spring-mass model equation without external force, so the behavior of the solution is the same as that of the spring-mass model. Therefore for small $\gamma > 0$ it exhibits a *damped oscillation*, i.e., it tends to zero in an oscillatory fashion.

In the case of a large angle rotation, the nonlinear model describes the motion. We note that we do not have an analytic method to find its solutions. Of course, numerical solution can be obtained easily. $\square$

Chapter 3
Systems of linear differential equations

In this chapter, we give a short introduction to the theory of the systems of linear differential equations. First, we start with an overview of the necessary notions and results from linear algebra.

3.1 Background from linear algebra

Let $\mathbf{A}$ be an $n \times n$ real matrix and $\mathbf{I}$ be the $n \times n$ identity matrix. The nth-order polynomial

$$p(\lambda) := \det(\mathbf{A} - \lambda \mathbf{I})$$

is called the *characteristic polynomial* of matrix $\mathbf{A}$, the roots of p are called *eigenvalues* of $\mathbf{A}$, and the non-zero solutions of the equation

$$\mathbf{A}\boldsymbol{\xi} = \lambda \boldsymbol{\xi}$$

or equivalently the equation

$$(\mathbf{A} - \lambda \mathbf{I})\boldsymbol{\xi} = \mathbf{0}$$

are called the *eigenvectors* of the matrix $\mathbf{A}$. If λ is a multiple root of order k of the characteristic polynomial p, then we say that the *algebraic multiplicity* of λ is k.

We summarize some known properties of the eigenvalues and eigenvectors in the next theorem.

Theorem 3.1 *Let $\mathbf{A}$ be an $n \times n$ real matrix.*

(i) The eigenvectors corresponding to an eigenvalue λ of $\mathbf{A}$ form a linear subspace of $\mathbb{C}^n$.

(ii) If λ is a real eigenvalue of A, then the corresponding eigenvector $\boldsymbol{\xi}$ can be real too.

(iii) If $\lambda_1, \ldots, \lambda_s$ are pairwise distinct eigenvalues of $\mathbf{A}$, then the corresponding eigenvectors $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(s)}$ are linearly independent.

(iv) If $\mathbf{A}$ is symmetric, then its has n linearly independent eigenvectors.

(v) If $\mathbf{A}$ has a complex eigenvalue $\lambda = \alpha + i\beta$ with a corresponding eigenvector $\boldsymbol{\xi} = \mathbf{u} + i\mathbf{v}$ ($\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$) then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

(vi) If $\lambda = \alpha + i\beta$ is a complex eigenvalue of $\mathbf{A}$ with a corresponding eigenvector $\boldsymbol{\xi} = \mathbf{u} + i\mathbf{v}$ then $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue of $\mathbf{A}$ with a corresponding eigenvector $\bar{\boldsymbol{\xi}} = \mathbf{u} - i\mathbf{v}$.

The set of eigenvectors corresponding to a fixed eigenvalue λ is called the *eigenspace* associated to λ. The dimension of the eigenspace associated to λ is called the *geometric multiplicity* of λ. It is known that the geometric multiplicity is always less or equal to the algebraic multiplicity of λ. It follows from part (iv) of the previous theorem that for symmetric matrices the geometric multiplicity is always equal to the algebraic multiplicity of any eigenvalue. We will see examples later that for non-symmetric matrices the geometric multiplicity can be strictly less than the algebraic multiplicity.

Example 3.2 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ -3 & -5 \end{pmatrix}.$$

Its characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} 2-\lambda & 2 \\ -3 & -5-\lambda \end{pmatrix} = \lambda^2 + 3\lambda - 4,$$

hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -4$ and $\lambda_2 = 1$. Clearly, the algebraic multiplicity of both eigenvalues are 1, and so the geometric multiplicities are also equal to 1.

Consider the eigenvector equation

$$\begin{pmatrix} 2-\lambda & 2 \\ -3 & -5-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

First consider the eigenvalue $\lambda_1 = -4$. Substituting to this value to the above equation we get

$$\begin{aligned} 6\xi_1 + 2\xi_2 &= 0 \\ -3\xi_1 - \xi_2 &= 0. \end{aligned}$$

The two equations are dependent, so we can omit any of the equations. We consider, e.g., $-3\xi_1 - \xi_2 = 0$, which has infinitely many solutions $\xi_2 = -3\xi_1$. One possible solution is

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Now consider the second eigenvalue $\lambda_2 = 1$. Then the eigenvector equations are

$$\begin{aligned} \xi_1 + 2\xi_2 &= 0 \\ -3\xi_1 - 6\xi_2 &= 0, \end{aligned}$$

which gives, e.g.,

$$\boldsymbol{\xi}^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

□

3.2 Linear systems of differential equations

Let $I \subset \mathbb{R}$ be an open interval, $t_0 \in I$, $a_{ij}, f_j \colon I \to \mathbb{R}$ $(i, j = 1, \ldots, n)$ functions, and consider the n-dimensional systems of linear differential equations

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\ &\vdots \\ x_n'(t) &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t) \end{aligned}$$

and the corresponding IC

$$x_1(t_0) = z_1, \quad \ldots, \quad x_n(t_0) = z_n.$$

We can rewrite the above problem in a vectorial form

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t), \qquad t \in I, \tag{3.1}$$

and

$$\mathbf{x}(t_0) = \mathbf{z}, \tag{3.2}$$

where $\mathbf{A} \colon I \to \mathbb{R}^{n\times n}$, $\mathbf{f} \colon I \to \mathbb{R}^n$,

$$\mathbf{A}(t) = (a_{ij}(t))_{n\times n}, \quad \mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^T, \quad \mathbf{f}(t) = (f_1(t), \ldots, f_n(t))^T, \quad \mathbf{z} = (z_1, \ldots, z_n)^T.$$

We will assume that $\mathbf{A}$ and $\mathbf{f}$ are continuous functions.

Consider the associated homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \qquad t \in I. \tag{3.3}$$

Applying Theorem 1.25 for this problem, we get the following result.

Theorem 3.3 *Suppose $\mathbf{A}\colon I \to \mathbb{R}^{n\times n}$ and $\mathbf{f}\colon I \to \mathbb{R}^n$ are continuous functions. Then the IVP (3.1)-(3.2) has a unique solution on the interval I for all $\mathbf{z} \in \mathbb{R}^n$.*

Just like for scalar linear homogeneous equations, any linear combination of two solutions is also a solution of the equation.

Theorem 3.4 *The set of solutions of the homogeneous Eq. (3.3) form a vector space.*

The set $\{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\}$ is called the *fundamental set of solutions* of the homogeneous Eq. (3.3) on the interval I, if $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are solutions of Eq. (3.3) and they are linearly independent on the interval I.

We put the formula of the vector function $\mathbf{x}^{(1)}$ to the first column of a matrix, the formula of the vector function $\mathbf{x}^{(2)}$ to the second column, and so on, the formula of the vector function $\mathbf{x}^{(n)}$ to the nth column. The resulting matrix is denoted by $(\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t))$. Its determinant is called the *Wronskian* of the solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$:

$$W(t) = \det(\mathbf{x}^{(1)}(t), \dots, \mathbf{x}^{(n)}(t)).$$

The following results can be proved.

Theorem 3.5 *The vector valued functions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent on the interval I, if and only if their Wronskian is not identically equal to 0 on the interval I.*

Theorem 3.6 *Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be solutions of the Eq. (3.3) corresponding to the IC*

$$\mathbf{x}^{(1)}(t_0) = \mathbf{z}^{(1)}, \quad \dots, \quad \mathbf{x}^{(n)}(t_0) = \mathbf{z}^{(n)}, \tag{3.4}$$

respectively. Then the functions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent on I if and only if the vectors

$$\mathbf{z}^{(1)}, \dots, \mathbf{z}^{(n)}$$

are linearly independent, i.e.

$$W(t_0) \neq 0.$$

Theorem 3.7 *The set of solutions of the homogeneous Eq. (3.3) is an n-dimensional vector space.*

3.3 Homogeneous linear systems with constant coefficients

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, and consider the *homogeneous linear system with constant coefficients*

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad t \in \mathbb{R}. \tag{3.1}$$

We are looking for solutions in the form

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\boldsymbol{\xi},$$

where $\boldsymbol{\xi}$ is a real or complex vector, λ is a real or complex constant. Then

$$\mathbf{x}'(t) = \lambda \mathrm{e}^{\lambda t}\boldsymbol{\xi},$$

so substituting into Eq. (3.1) we get

$$\lambda \mathrm{e}^{\lambda t}\boldsymbol{\xi} = \mathbf{A}\mathrm{e}^{\lambda t}\boldsymbol{\xi}.$$

This equation holds if

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = \mathbf{0},$$

i.e., λ is an eigenvalue of the matrix $\mathbf{A}$, and $\boldsymbol{\xi}$ is the eigenvector corresponding to λ.

According to Theorem 3.7, it is enough to find n linearly independent solutions, since then their linear combinations give all solutions of Eq. (3.1). We consider several cases.

Case 1: pairwise distinct eigenvalues

Suppose $\lambda_1, \dots, \lambda_n$ are pairwise distinct eigenvalues of the matrix A, and $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ are the eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$, respectively. It is known from linear algebra (see Theorem 3.1) that the vectors $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ are linearly independent. Then the functions

$$\mathbf{x}^{(1)}(t) = \mathrm{e}^{\lambda_1 t}\boldsymbol{\xi}^{(1)}, \quad \dots, \quad \mathbf{x}^{(n)}(t) = \mathrm{e}^{\lambda_n t}\boldsymbol{\xi}^{(n)}, \qquad t \in \mathbb{R} \tag{3.2}$$

are solutions of the homogeneous Eq. (3.1). On the other hand,

$$W(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})(0) = \det(\mathbf{x}^{(1)}(0), \dots, \mathbf{x}^{(n)}(0)) = \det(\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}) \neq 0,$$

since $\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(n)}$ are linearly independent. Therefore the solutions $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ are linearly independent, hence (3.2) forms a fundamental set of solutions of Eq. (3.1). Therefore the general solution of Eq. (3.1) is

$$\mathbf{x}(t) = c_1\mathrm{e}^{\lambda_1 t}\boldsymbol{\xi}^{(1)} + \cdots + c_n\mathrm{e}^{\lambda_n t}\boldsymbol{\xi}^{(n)}. \tag{3.3}$$

If all eigenvalues are real, the fundamental set of solutions of Eq. (3.2) consists of real functions, but if the matrix $\mathbf{A}$ has complex eigenvalues, then some of the functions in (3.2) are complex valued. In the next case we will show that these complex solutions can be replaced by real solutions.

Example 3.8 Solve the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ -3 & -5 \end{pmatrix}\mathbf{x} \tag{3.4}$$

corresponding to the IC

$$\mathbf{x}(0) = \begin{pmatrix} -5 \\ 0 \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are $\lambda_1 = -4$ and $\lambda_2 = 1$, and the associated eigenvectors are $\boldsymbol{\xi}^{(1)} = (1, -3)^T$ and $\boldsymbol{\xi}^{(2)} = (-2, 1)^T$, respectively. Therefore the general solution is

$$\mathbf{x}(t) = c_1\mathrm{e}^{-4t}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + c_2\mathrm{e}^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \tag{3.5}$$

so its component functions are

$$\begin{aligned} x_1(t) &= c_1 e^{-4t} - 2c_2 e^t \\ x_2(t) &= -3c_1 e^{-4t} + c_2 e^t. \end{aligned}$$

The IC yields

$$\begin{array}{rcrcr} c_1 & - & 2c_2 & = & -5 \\ -3c_1 & + & c_2 & = & 0, \end{array}$$

which gives $c_1 = 1$ and $c_2 = 3$. Therefore the solution of the IVP is

$$\mathbf{x}(t) = e^{-4t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} + 3e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \tag{3.6}$$

and hence its components are

$$\begin{aligned} x_1(t) &= e^{-4t} - 6e^t \\ x_2(t) &= -3e^{-4t} + 3e^t. \end{aligned}$$

□

Case 2: complex eigenvalues

Suppose $\lambda = \alpha + i\beta$ is a complex eigenvalue of $\mathbf{A}$, and

$$\boldsymbol{\xi} = \mathbf{u} + i\mathbf{v}$$

is the corresponding eigenvector. (Here $\mathbf{u}$ and $\mathbf{v}$ are the real and imaginary parts of the complex vector $\boldsymbol{\xi}$.) Then it is known that $\bar{\lambda} = \alpha - i\beta$ is also an eigenvalue of $\mathbf{A}$, and the corresponding eigenvector is

$$\bar{\boldsymbol{\xi}} = \mathbf{u} - i\mathbf{v}.$$

Then

$$\begin{aligned} \mathbf{x}(t) &= e^{\lambda t}\boldsymbol{\xi} \\ &= e^{(\alpha+i\beta)t}(\mathbf{u} + i\mathbf{v}) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{u} + i\mathbf{v}) \\ &= e^{\alpha t}\Big(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t + i(\mathbf{v}\cos\beta t + \mathbf{u}\sin\beta t)\Big) \end{aligned}$$

is a complex valued solution of Eq. (3.1). Like in scalar equations, it can be shown that the real and imaginary parts of a complex solution are also solutions of Eq. (3.1). Therefore

$$\mathbf{x}^{(1)}(t) = e^{\alpha t}(\mathbf{u}\cos\beta t - \mathbf{v}\sin\beta t) \qquad \text{and} \qquad \mathbf{x}^{(2)}(t) = e^{\alpha t}(\mathbf{v}\cos\beta t + \mathbf{u}\sin\beta t)$$

are both solutions, and it can be shown that the vectors $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are always linearly independent. Hence in the formula of the general solution (3.3), the complex solutions $e^{\lambda t}\boldsymbol{\xi}$ and $e^{\bar{\lambda} t}\bar{\boldsymbol{\xi}}$ can be replaced by the real solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

Example 3.9 Solve the IVP

$$\mathbf{x}' = \begin{pmatrix} -9 & 4 \\ -10 & 3 \end{pmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

The eigenvalues of the coefficient matrix are $\lambda_1 = -3+2i$ and $\lambda_2 = -3-2i$, and the corresponding eigenvectors are $\boldsymbol{\xi}^{(1)} = (2, 3+i)^T$ and $\boldsymbol{\xi}^{(2)} = (2, 3-i)^T$. Therefore the complex solution is

$$\begin{aligned} e^{(-3+2i)t}\begin{pmatrix} 2 \\ 3+i \end{pmatrix} &= e^{-3t}(\cos 2t + i\sin 2t)\begin{pmatrix} 2 \\ 3+i \end{pmatrix} \\ &= e^{-3t}\begin{pmatrix} 2\cos 2t + 2i\sin 2t \\ 3\cos 2t - \sin 2t + i(\cos 2t + 3\sin 2t) \end{pmatrix}. \end{aligned}$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-3t} \begin{pmatrix} 2\cos 2t \\ 3\cos 2t - \sin 2t \end{pmatrix} + c_2 \mathrm{e}^{-3t} \begin{pmatrix} 2\sin 2t \\ \cos 2t + 3\sin 2t \end{pmatrix}.$$

The IC yields

$$\begin{array}{rcrcr} 2c_1 & & & = & -2 \\ 3c_1 & + & c_2 & = & 1, \end{array}$$

and so $c_1 = -1$ and $c_2 = 4$. Hence the solution of the IVP is

$$\mathbf{x}(t) = -\mathrm{e}^{-3t} \begin{pmatrix} 2\cos 2t \\ 3\cos 2t - \sin 2t \end{pmatrix} + 4\mathrm{e}^{-3t} \begin{pmatrix} 2\sin 2t \\ \cos 2t + 3\sin 2t \end{pmatrix} = \mathrm{e}^{-3t} \begin{pmatrix} -2\cos 2t + 8\sin 2t \\ \cos 2t + 13\sin 2t \end{pmatrix}.$$

□

Case 3/a: multiple eigenvalues

We consider the case of a multiple eigenvalue in the 3-dimensional case. The general case is similar. First, we suppose that λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity 3, and $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \boldsymbol{\xi}^{(3)}$ are linearly independent eigenvectors to λ. Then

$$\mathbf{x}^{(1)}(t) = \mathrm{e}^{\lambda t}\boldsymbol{\xi}^{(1)}, \quad \mathbf{x}^{(2)}(t) = \mathrm{e}^{\lambda t}\boldsymbol{\xi}^{(2)}, \quad \mathbf{x}^{(3)}(t) = \mathrm{e}^{\lambda t}\boldsymbol{\xi}^{(3)}$$

are three linearly independent solutions, since the vectors $\mathbf{x}^{(1)}(0) = \boldsymbol{\xi}^{(1)}$, $\mathbf{x}^{(2)}(0) = \boldsymbol{\xi}^{(2)}$, $\mathbf{x}^{(3)}(0) = \boldsymbol{\xi}^{(3)}$ are linearly independent. Then the general solution is

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\Big(c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)} + c_3\boldsymbol{\xi}^{(3)}\Big).$$

In the second case, we suppose the eigenvalues of the matrix $\mathbf{A}$ are λ_1 and λ_2, where λ_1 is a double, λ_2 is a single eigenvalue, and $\boldsymbol{\xi}^{(11)}$ and $\boldsymbol{\xi}^{(12)}$ are two linearly independent eigenvectors corresponding to λ_1. Let $\boldsymbol{\xi}^{(2)}$ be the eigenvector corresponding to λ_2. Then it is easy to see that the general solution is

$$\mathbf{x}(t) = \mathrm{e}^{\lambda_1 t}\Big(c_1\boldsymbol{\xi}^{(11)} + c_2\boldsymbol{\xi}^{(12)}\Big) + c_3\mathrm{e}^{\lambda_2 t}\boldsymbol{\xi}^{(2)}.$$

Example 3.10 Solve the linear system

$$\mathbf{x}' = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & -1 \\ -2 & 2 & -3 \end{pmatrix} \mathbf{x}.$$

The characteristic polynomial of the coefficient matrix is

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -\lambda & -2 & 1 \\ -2 & -\lambda & -1 \\ -2 & 2 & -3-\lambda \end{pmatrix} \\ &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -\lambda^3 + \lambda^2 - 4\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2. \end{aligned}$$

So the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = -2$, where the algebraic multiplicity of λ_1 is 1, and that of λ_2 is 2.

Consider first $\lambda_1 = 1$. The corresponding eigenvector equation is

$$\begin{aligned} -\xi_1 - 2\xi_2 + \xi_3 &= 0 \\ -2\xi_1 - \xi_2 - \xi_3 &= 0 \\ -2\xi_1 + 2\xi_2 - 4\xi_3 &= 0. \end{aligned}$$

If we multiply the first equation by -1 and add it to the second equation, and we multiply the resulting equation by 2 we get the third equation. So we can omit one equation, e.g., the third one. Adding the remaining two equations we get that $\xi_1 = -\xi_2$. So if we set $\xi_1 = 1$ then we have $\xi_2 = -1$ and therefore we get $\xi_3 = -1$. Hence

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 1$.

Now consider $\lambda_2 = -2$. Then the eigenvector equations are

$$\begin{aligned} 2\xi_1 - 2\xi_2 + \xi_3 &= 0 \\ -2\xi_1 + 2\xi_2 - \xi_3 &= 0 \\ -2\xi_1 + 2\xi_2 - \xi_3 &= 0. \end{aligned}$$

Clearly, the second and third equations can be omitted, so only one equation remains:

$$2\xi_1 - 2\xi_2 + \xi_3 = 0.$$

We can set, e.g., the values of ξ_1 and ξ_2 independently, hence we get easily that the vectors

$$\boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

are both eigenvectors corresponding to λ_2, and they are linearly independent. Therefore the eigenspace of λ_2 is two-dimensional, so the geometric multiplicity of $\mathbf{A}$ is 2. Therefore the general solution of the equation is

$$\mathbf{x}(t) = c_1 \mathrm{e}^t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + c_2 \mathrm{e}^{-2t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + c_3 \mathrm{e}^{-2t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

□

In the next example, we show a two-dimensional system, where the geometric multiplicity of the eigenvalue is two, i.e., there are two linearly independent eigenvectors.

Example 3.11 Consider the system

$$\mathbf{x}' = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}.$$

It is easy to check that the eigenvalue of the coefficient matrix is $\lambda = -3$, which has algebraic multiplicity 2. The geometric multiplicity is also 2, since, e.g., $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors. Therefore, the general solution of the system is

$$\mathbf{x}(t) = \mathrm{e}^{-3t} \left[c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \mathrm{e}^{-3t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

□

Case 3/b: multiple eigenvalues

Suppose again that the algebraic multiplicity of the eigenvalue λ is at least 2, but the geometric multiplicity is less than the algebraic multiplicity. Let $\boldsymbol{\xi}$ be an eigenvector corresponding to λ. Then $\mathrm{e}^{\lambda t}\boldsymbol{\xi}$ is a solution, but we need one more solution corresponding to λ, which is linearly independent from the first solution.

Analogously to the scalar case, we are looking for a second solution in the form

$$t\mathrm{e}^{\lambda t}\boldsymbol{\xi} + \mathrm{e}^{\lambda t}\boldsymbol{\eta}. \tag{3.7}$$

Substituting into the Eq. (3.1) we get

$$\mathrm{e}^{\lambda t}\boldsymbol{\xi} + \lambda t\mathrm{e}^{\lambda t}\boldsymbol{\xi} + \lambda\mathrm{e}^{\lambda t}\boldsymbol{\eta} = \mathbf{A}t\mathrm{e}^{\lambda t}\boldsymbol{\xi} + \mathbf{A}\mathrm{e}^{\lambda t}\boldsymbol{\eta}.$$

This yields a solution if the coefficients of the functions $t\mathrm{e}^{\lambda t}$ and $\mathrm{e}^{\lambda t}$ are identical on both sides of the equation:

$$\begin{aligned} \lambda\boldsymbol{\xi} &= \mathbf{A}\boldsymbol{\xi} \\ \boldsymbol{\xi} + \lambda\boldsymbol{\eta} &= \mathbf{A}\boldsymbol{\eta}, \end{aligned}$$

or in an equivalent form,

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \tag{3.8}$$

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{3.9}$$

Then Eq. (3.8) means that $\boldsymbol{\xi}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. The solution $\boldsymbol{\eta}$ of the Eq. (3.9) is called the *generalized eigenvector* of $\mathbf{A}$ corresponding to the eigenvalue λ. Clearly, the generalized eigenvector does not belong to the eigenspace of λ, so it is always linearly independent of $\boldsymbol{\xi}$. The following theorem holds.

Theorem 3.12 *Let λ be an eigenvalue of $\mathbf{A}$ with geometric multiplicity less than its algebraic multiplicity. Then Eq. (3.9) has at least one solution $\boldsymbol{\eta}$, which does not belong to an eigenspace corresponding to any eigenvalue.*

Example 3.13 Consider the system

$$\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{x}.$$

Its characteristic polynomial is

$$p(\lambda) = \det\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2,$$

hence the algebraic multiplicity of the eigenvalue $\lambda = -2$ is 2.

Consider the eigenvector equation

$$\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $\lambda = -2$ into the above equation, we get

$$\begin{aligned} -\xi_1 + \xi_2 &= 0 \\ -\xi_1 + \xi_2 &= 0. \end{aligned}$$

One possible solution is

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and we can observe that all other solutions are constant multiples of this solution, hence the geometric multiplicity of $\lambda = -2$ is 1.

We need to find a generalized eigenvector. Consider the generalized eigenvector equation

$$\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

Substituting λ and $\boldsymbol{\xi}$, we have

$$\begin{aligned} -\eta_1 + \eta_2 &= 1 \\ -\eta_1 + \eta_2 &= 1. \end{aligned}$$

The two equations are identical, so we omit one, and we get infinitely many solution for $\boldsymbol{\eta}$. One possible solution is

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore (3.7) gives the solution

$$t\mathrm{e}^{\lambda t}\boldsymbol{\xi} + \mathrm{e}^{\lambda t}\boldsymbol{\eta} = t\mathrm{e}^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathrm{e}^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{e}^{-2t}\begin{pmatrix} t \\ t+1 \end{pmatrix}.$$

Clearly, this is independent of the first solution, hence the general solution of the system is

$$\mathbf{x}(t) = c_1\mathrm{e}^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\mathrm{e}^{-2t}\begin{pmatrix} t \\ t+1 \end{pmatrix} = \mathrm{e}^{-2t}\begin{pmatrix} c_1 + c_2 t \\ c_1 + c_2(t+1) \end{pmatrix}.$$

□

Example 3.14 Solve the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -6 & 0 \\ -10 & 7 & -8 \\ -8 & 8 & -7 \end{pmatrix}\mathbf{x}.$$

Consider the characteristic polynomial of the coefficient matrix

$$p(\lambda) = \det\begin{pmatrix} 3-\lambda & -6 & 0 \\ -10 & 7-\lambda & -8 \\ -8 & 8 & -7-\lambda \end{pmatrix}$$

We expand it with respect to the third column:

$$\begin{aligned} p(\lambda) &= 8 \cdot \det\begin{pmatrix} 3-\lambda & -6 \\ -8 & 8 \end{pmatrix} + (7+\lambda)\cdot\det\begin{pmatrix} 3-\lambda & -6 \\ -10 & 7-\lambda \end{pmatrix} \\ &= 8(-8\lambda - 24) + (-7-\lambda)(\lambda^2 - 10\lambda - 39) \\ &= -64(\lambda+3) - (7+\lambda)(\lambda+3)(\lambda-13) \\ &= -(\lambda+3)[64 + (7+\lambda)(\lambda-13)] \\ &= -(\lambda+3)[\lambda^2 - 6\lambda - 27)] \\ &= -(\lambda+3)^2(\lambda-9). \end{aligned}$$

This shows that $\lambda_1 = 9$ is a single, $\lambda_2 = -3$ is a double eigenvalue. To compute the eigenvector corresponding to $\lambda_1 = 9$, we consider

$$\begin{aligned} -6\xi_1 - 6\xi_2 \quad &= 0 \\ -10\xi_1 - 2\xi_2 - 8\xi_3 &= 0 \\ -8\xi_1 + 8\xi_2 - 16\xi_3 &= 0. \end{aligned}$$

If we multiply the first equation by -2, the second by 2 and we add the equations, we get the third equation. This means that we can omit the third equation. Then we get

$$\begin{aligned} -6\xi_1 - 6\xi_2 \quad &= 0 \\ -10\xi_1 - 2\xi_2 - 8\xi_3 &= 0. \end{aligned}$$

One possible solution is $\boldsymbol{\xi}^{(1)} = (-1, 1, 1)^T$.

Now consider $\lambda_2 = -3$. We get the corresponding eigenvector equations

$$\begin{aligned} 6\xi_1 - 6\xi_2 \qquad &= 0 \\ -10\xi_1 + 10\xi_2 - 8\xi_3 &= 0 \\ -8\xi_1 + 8\xi_2 - 4\xi_3 &= 0. \end{aligned}$$

Twice the third equation plus the first one gives the second equation. Therefore we can omit the second equation:

$$\begin{aligned} 6\xi_1 - 6\xi_2 \qquad &= 0 \\ -8\xi_1 + 8\xi_2 - 4\xi_3 &= 0. \end{aligned}$$

So the geometric multiplicity of $\lambda_2 = -3$ is 1, and one possible corresponding eigenvector is $\boldsymbol{\xi}^{(2)} = (1, 1, 0)^T$.

To write down the third solution, we need to find the generalized eigenvector corresponding to $\lambda_2 = 3$. Eq. (3.9) with $\boldsymbol{\xi} = \boldsymbol{\xi}^{(2)}$ yields

$$\begin{aligned} 6\eta_1 - 6\eta_2 \qquad &= 1 \\ -10\eta_1 + 10\eta_2 - 8\eta_3 &= 1 \\ -8\eta_1 + 8\eta_2 - 4\eta_3 &= 0. \end{aligned}$$

The second equation can be omitted, since twice the third equations plus the first equation equals to the second one:

$$\begin{aligned} 6\eta_1 - 6\eta_2 \qquad &= 1 \\ -8\eta_1 + 8\eta_2 - 4\eta_3 &= 0. \end{aligned}$$

The remaining two equations are independent. For example set $\eta_1 = 1$, then then $\eta_2 = 0$ and $\eta_3 = -2$, hence

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 \mathrm{e}^{9t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \left[t\mathrm{e}^{-3t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mathrm{e}^{-3t} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right].$$

□

Example 3.15 Consider the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix} \mathbf{x}.$$

The characteristic polynomial of the coefficient matrix is

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} 3-\lambda & -1 & -2 \\ -1 & 3-\lambda & 2 \\ 1 & -1 & -\lambda \end{pmatrix} \\ &= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 \\ &= -(\lambda - 2)^3. \end{aligned}$$

Hence $\lambda = 2$ is a triple eigenvalue of $\mathbf{A}$. The eigenvector equations are

$$\xi_1 - \xi_2 - 2\xi_3 = 0$$

$$\begin{aligned} -\xi_1 + \xi_2 + 2\xi_3 &= 0 \\ \xi_1 - \xi_2 - 2\xi_3 &= 0. \end{aligned}$$

Here two equations can be omitted, so we get

$$\xi_1 - \xi_2 - 2\xi_3 = 0.$$

Here we can find two linearly independent solutions. For example

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

are two eigenvectors. We need the generalized eigenvector:

$$\begin{aligned} \eta_1 - \eta_2 - 2\eta_3 &= \xi_1 \\ -\eta_1 + \eta_2 + 2\eta_3 &= \xi_2 \\ \eta_1 - \eta_2 - 2\eta_3 &= \xi_3, \end{aligned}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T$ is an eigenvector of the coefficient matrix. We can see that neither $\boldsymbol{\xi}^{(1)}$ nor $\boldsymbol{\xi}^{(2)}$ can be used, since the above system has no solution. We look for the right hand side in the form $\boldsymbol{\xi} = c_1\boldsymbol{\xi}^{(1)} + c_2\boldsymbol{\xi}^{(2)}$, where the scalars c_1 and c_2 are to be determined. Then we get

$$\begin{aligned} \eta_1 + \eta_2 + 2\eta_3 &= 2c_1 + c_2 \\ \eta_1 + \eta_2 + 2\eta_3 &= c_2 \\ -\eta_1 - \eta_2 - 2\eta_3 &= c_1, \end{aligned}$$

which has a solution if and only if

$$2c_1 + c_2 = c_1 \qquad \text{and} \qquad c_1 = -c_2,$$

for example, $c_1 = 1$ and $c_2 = -1$. Then

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

so the generalized eigenvector equations are

$$\begin{aligned} \eta_1 - \eta_2 - 2\eta_3 &= 1 \\ -\eta_1 + \eta_2 + 2\eta_3 &= -1 \\ \eta_1 - \eta_2 - 2\eta_3 &= 1. \end{aligned}$$

Simplifying it we get

$$\eta_1 - \eta_2 - 2\eta_3 = 1,$$

which gives, e.g.,

$$\boldsymbol{\eta} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore the general solution of the system is

$$\mathbf{x}(t) = \mathrm{e}^{2t} \left\{ c_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \left[t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \right\}.$$

□

Case 3/c: multiple eigenvalues

Suppose $\mathbf{A}$ is a 3×3 matrix, where its eigenvalues λ has an algebraic multiplicity 3, but its geometric multiplicity is only 1. Let $\boldsymbol{\xi}$ be a corresponding eigenvector. Then the generalized eigenvector equation (3.9) has a solution $\boldsymbol{\eta}$, but suppose there is no other generalized eigenvector linearly independent from $\boldsymbol{\eta}$. Then $e^{\lambda t}\boldsymbol{\xi}$ and $te^{\lambda t}\boldsymbol{\xi} + e^{\lambda t}\boldsymbol{\eta}$ are two linearly independent solutions of Eq. (3.1), but we need a third solution which is linearly independent from the above two solutions. We look for the third solution in the form

$$\mathbf{x}(t) = \frac{1}{2}t^2 e^{\lambda t}\boldsymbol{\xi} + te^{\lambda t}\boldsymbol{\eta} + e^{\lambda t}\boldsymbol{\omega}.$$

We can easily check that the above function $\mathbf{x}(t)$ is a solution of Eq. (3.1) if and only if

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} &= \mathbf{0}, \\ (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} &= \boldsymbol{\xi}, \\ (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\omega} &= \boldsymbol{\eta}. \end{aligned}$$

Such a vector $\boldsymbol{\omega}$ is called the *generalized eigenvector of rank 3.* The eigenvector $\boldsymbol{\xi}$ is a generalized eigenvector of rank 1, the vector $\boldsymbol{\eta}$ is called a generalized eigenvector of rank 2.

Example 3.16 Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix} \mathbf{x}.$$

Compute the eigenvalues of the coefficient matrix:

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -1-\lambda & -1 & -1 \\ 1 & -2-\lambda & -1 \\ -1 & 1 & -\lambda \end{pmatrix} \\ &= -\lambda^3 - 3\lambda^2 - 3\lambda - 1 \\ &= -(\lambda+1)^3, \end{aligned}$$

So $\lambda = -1$ is a triple eigenvalue of the matrix. The eigenvector equation is

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

hence

$$\begin{aligned} -\xi_2 - \xi_3 &= 0 \\ \xi_1 - \xi_2 - \xi_3 &= 0. \end{aligned}$$

For example a corresponding eigenvector is

$$\boldsymbol{\xi} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

and the geometric multiplicity is only 1. The generalized eigenvector equation is

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

and hence

$$\begin{aligned} -\eta_2 - \eta_3 &= 0 \\ \eta_1 - \eta_2 - \eta_3 &= -1 \\ -\eta_1 + \eta_2 + \eta_3 &= 1. \end{aligned}$$

The third equation can be omitted, but the first two equations are independent.

$$\begin{aligned} -\eta_2 - \eta_3 &= 0 \\ \eta_1 - \eta_2 - \eta_3 &= -1. \end{aligned}$$

One possible solution is $\eta_1 = -1$, $\eta_2 = 1$ and $\eta_3 = -1$, hence

$$\boldsymbol{\eta} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix},$$

but there is no other linearly independent solution of the equation. We need to find a generalized eigenvector of rank 3:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

The third equation can be omitted:

$$\begin{aligned} -\omega_2 - \omega_3 &= -1 \\ \omega_1 - \omega_2 - \omega_3 &= 1. \end{aligned}$$

One solution is

$$\boldsymbol{\omega} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

Hence the general solution of the system is

$$\mathbf{x}(t) = \mathrm{e}^{-t}\left\{ c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2\left[t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right] + c_3\left[\frac{t^2}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right]\right\}.$$

□

3.4 Applications

Example 3.17 Suppose given two tanks connected by two pipes (see Figure 3.1) containing salt solution. The first tank initially contains A_1 kg salt dissolved in V_1 l of solution, the second tank contains A_2 kg salt dissolved in V_2 l of solution. The solution is pumped from the first tank into the second tank with the rate of r l/min through the first pipe, and similarly, the fluid from the second tank is pumped into the first one through the second pipe with the velocity r l/min. We suppose that the volume of the solution in the pipes can be neglected, and the pipes are short, so we omit the time while the solution travels in the pipes. We also assume that the solution in both tanks are well-stirred, ie., the concentration in a tank is homogeneous. Compute the amount of the salt in both tanks at time t.

First, note that the volume of the solutions in both tanks and the total mass of salt in the tanks remain constant as time goes by. The mass of the salt in tank 1 and 2 at time t are denoted by $Q_1 = Q_1(t)$ and $Q_2 = Q_2(t)$, respectively. Then the concentration of the solutions in the tanks are Q_1/V_1 kg/l and Q_2/V_2 kg/l, respectively. The amount of salt in tank 1 is decreasing with a rate of rQ_1/V_1 kg/min because of the outflow from the tank, and it is increasing with the rate of rQ_2/V_2 kg/min because of the inflow into the tank. Therefore, the following equations describe the mass of salt in tanks

$$\begin{aligned} Q_1' &= -r\tfrac{Q_1}{V_1} + r\tfrac{Q_2}{V_2}, \qquad & Q_1(0) = A_1 \\ Q_2' &= r\tfrac{Q_1}{V_1} - r\tfrac{Q_2}{V_2}, \qquad & Q_2(0) = A_2. \end{aligned} \tag{3.1}$$

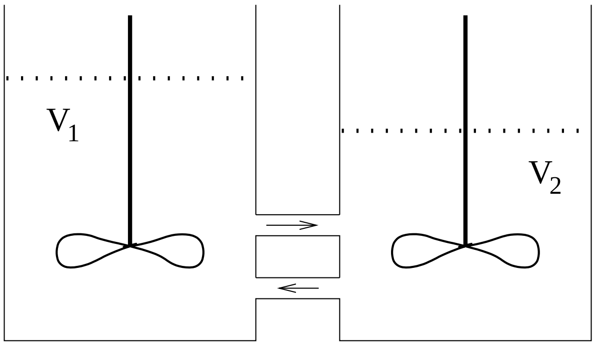

Figure 3.1: double tank

Consider the situation when initially the first tank contains 100 l of pure water, and the second tank contains 2 kg salt dissolved in 75 l solution. Suppose the solutions are pumped in pipes with a rate of 3 l/min. Then Eq. (3.1) has the form

$$\begin{aligned} Q_1' &= -\tfrac{3}{100}Q_1 + \tfrac{1}{25}Q_2, & Q_1(0) &= 0 \\ Q_2' &= \tfrac{3}{100}Q_1 - \tfrac{1}{25}Q_2, & Q_2(0) &= 2. \end{aligned}$$

The characteristic polynomial of the coefficient matrix is

$$\det\begin{pmatrix} -\frac{3}{100}-\lambda & \frac{1}{25} \\ \frac{3}{100} & -\frac{1}{25}-\lambda \end{pmatrix} = \left(\lambda+\frac{3}{100}\right)\left(\lambda+\frac{1}{25}\right) - \frac{3}{100}\frac{1}{25} = \lambda\left(\lambda+\frac{7}{100}\right).$$

This yields the eigenvalues $\lambda = 0, -\frac{7}{100}$, and the corresponding eigenvectors $(4,3)^T$ and $(-1,1)^T$. Therefore, the general solution of the system is

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = c_1\begin{pmatrix} 4 \\ 3 \end{pmatrix} + c_2 \mathrm{e}^{-\frac{7}{100}t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The initial conditions give

$$Q_1 = \frac{8}{7} - \frac{8}{7}\mathrm{e}^{-\frac{7}{100}t}, \qquad Q_2 = \frac{6}{7} + \frac{8}{7}\mathrm{e}^{-\frac{7}{100}t}.$$

□

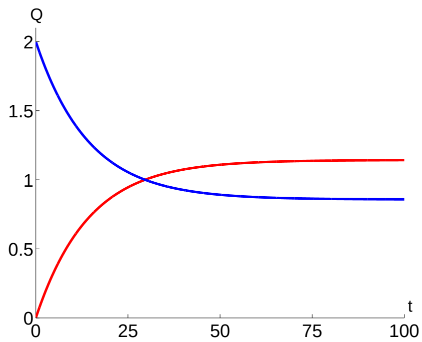

Figure 3.2: The mass of salt in tanks, red: $Q_1(t)$, blue: $Q_2(t)$

In the next examples, we show nonlinear population models.

Example 3.18 (predator-prey model) Suppose we have a population where there are two species: a predator and a prey. So we suppose that the first population preys upon the second one, but the second population lives on a different kind of food. An example is foxes and rabbits living in a same area, or a lake with two kind of fishes where the first species preys on the second one.

Let $x = x(t)$ and $y = y(t)$ denote the number or the size of the prey and the predator population at time t, respectively. We suppose that without the presence of predators the prey population increases ideally, i.e., it can be described by the Malthus-model with a growth rate $a > 0$ (in the environment food is sufficient for the preys). Similarly, the size of the predator population decreases exponentially without the prey population with a negative growth rate $-c$. On the other hand, the size of the prey population decreases when the predator population is present. It is natural to assume that the mortality rate is proportional to the number of encounters between the two species, so the rate of decrease is assumed to be $-bxy$. Similarly, the rate of increase of the predator population has the form dxy. We get therefore the *predator-prey model*

$$\begin{aligned} x' &= x(a - by) \\ y' &= y(-c + dx), \end{aligned} \tag{3.2}$$

which is a two-dimensional system of nonlinear differential equations. Here all parameters are positive: $a, b, c, d > 0$.

In Figure 3.3, we plotted the numerical solution of (3.2) corresponding to parameters $a = 2$, $b = 0.5$, $c = 1$, $d = 0.25$ and initial conditions $x(0) = 10$ and $y(0) = 6$. The blue curve represents the prey, the red curve represents the predator population. We can observe that the corresponding solutions are periodic. We can see that when the predator population is small, the size of the prey population is increasing, and vice versa, when the predator population is small, the prey population increases. □

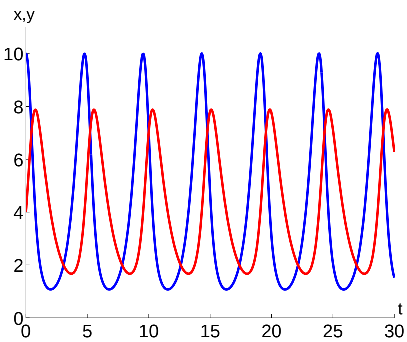

Figure 3.3: predator-prey model, blue: prey, red: predator population

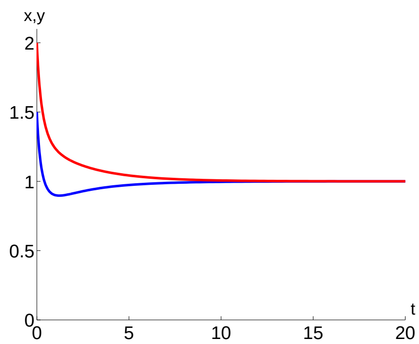

Figure 3.4: competing species, blue: $x(t)$, red: $y(t)$

Example 3.19 (competing species) Now we consider another situation of a two species model. We suppose that the two species are similar, both compete for the same food. We assume that in the absence of the other population the size of both species can be described by the logistic model, and the rate of change of each population decreases is proportional to the number of encounters of the two species. The resulting model has the form

$$\begin{aligned} x' &= x(a - bx - cy) \\ y' &= y(d - ex - fy), \end{aligned} \tag{3.3}$$

where $a, b, c, d, e, f > 0$.

In Figure 3.4, we plotted the numerical solution of (3.3) with $a = 3$, $b = 2$, $c = 1$, $d = 2$, $e = 1$, $f = 1$ and the initial conditions $x(0) = 1.5$ and $y(0) = 2$. The blue curve is the graph of x and the red curve is the graph of y. We observe that both solutions converge to a positive limit, which means that both species survive and after long time, there is balance of the two species. In Example 4.15 below, we examine again this particular system and we give a more complete analysis of the behavior of solutions. □

Chapter 4
Stability theory of differential equations

In this chapter, we investigate some basic geometric properties of autonomous systems, the notion of Liapunov stability, the classification of equilibriums of a planar linear system, the method of linearization and the Liapunov function technique.

4.1 Autonomous systems

The general form of a first-order explicit nonlinear system is

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}),$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. In many applications the variable t denotes time, so when the right-hand-side does not depend on t we call the equation *autonomous* or *time-independent* or *time-invariant*. Therefore, the general form of a first-order autonomous differential equation is

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \tag{4.1}$$

where $\mathbf{f} \colon \mathbb{R}^n \to \mathbb{R}^n$, or $\mathbf{f} \colon D \to \mathbb{R}^n$ and $D \subset \mathbb{R}^n$. It can be checked by substitution that if $\mathbf{y}(t)$ is a solution of Eq. (4.1), then the function $\mathbf{x}(t) = \mathbf{y}(t + \tau)$ is also a solution of Eq. (4.1) with arbitrary $\tau \in \mathbb{R}$. Therefore time translation of a solution of an autonomous equation is also a solution of the same equation. So without the loss of generality, we assume that the initial time is $t_0 = 0$, so the initial condition associated to Eq. (4.1) is

$$\mathbf{x}(0) = \mathbf{z}. \tag{4.2}$$

We assume throughout this chapter that the IVP (4.1)-(4.2) has a unique solution corresponding to any $\mathbf{z} \in \mathbb{R}^n$, and this solution is denoted by $\mathbf{x}(t; \mathbf{z})$.

A constant solution $\mathbf{x}(t) = \mathbf{u}$ of Eq. (4.1) is called *equilibrium solution*, and $\mathbf{u}$ is called *equilibrium point* or *critical point* or just simply *equilibrium.* The derivative of a constant solution is constant 0, so the equilibrium points are the roots of the function $\mathbf{f}$.

Theorem 4.1 *The vector* $\mathbf{u}$ *is an equilibrium point of Eq. (4.1) if and only if* $\mathbf{f}(\mathbf{u}) = \mathbf{0}$.

An *integral curve* or a *solution curve* of a solution $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^T, \quad t \in I$ is the curve

$$\{(t, \mathbf{x}(t)) \colon t \in I\} \subset \mathbb{R} \times \mathbb{R}^n,$$

i.e., the graph of the solution. In practice we usually draw the ith component integral curves $\{(t, x_i(t)) \colon t \in I\}$ $(i = 1, \ldots, n)$.

For 2 or 3-dimensional systems there is an other frequently used illustration of the solutions. We call the curve

$$\{\mathbf{x}(t) \colon t \in I\} \subset \mathbb{R}^n$$

as the *trajectory* or *orbit* of the solution, the space $\mathbb{R}^n$ is called the *phase space* (in the 2-dimensional case *phase plane*). A set of trajectories of a differential equation is called *phase portrait* of the equation. Typically, we draw the trajectories as directed curves, since we are interested in seeing the path of the solution as time increases.

Example 4.2 Consider the second-order linear homogeneous scalar equation

$$v'' + v = 0.$$

It is easy to see that its general solution is $v(t) = c_1 \cos t + c_2 \sin t$. We can rewrite the second-order equation in a system form by introducing the variables $x = v$, $y = v'$ and $\mathbf{x} = (x, y)^T$:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

We solve this system using the ICs

$$\mathbf{x}(0) = (0.5, 0)^T, \quad \mathbf{x}(0) = (1, 0)^T, \quad \mathbf{x}(0) = (0, 1.5)^T, \quad \mathbf{x}(0) = (0, 2)^T,$$

respectively. The corresponding component-wise integral curves can be seen in Figures 4.1 and 4.2. For example the solution corresponding to $\mathbf{x}(0) = (1, 0)^T$ is $\mathbf{x}(t) = (\cos t, -\sin t)^T$. Its trajectory is

$$x = \cos t, \qquad y = -\sin t,$$

which yields a circle centered at the origin with radius 1, since

$$x^2 + y^2 = 1$$

is satisfied along the solution. For the general solution $x = c_1 \cos t + c_2 \sin t$ and $y = -c_1 \sin t + c_2 \cos t$ we get

$$\begin{aligned} x^2 + y^2 &= (c_1 \cos t + c_2 \sin t)^2 + (-c_1 \sin t + c_2 \cos t)^2 \\ &= c_1^2 \cos^2 t + c_2^2 \sin^2 t + 2c_1 c_2 \cos t \sin t + c_1^2 \sin^2 t + c_2^2 \cos^2 t - 2c_1 c_2 \cos t \sin t \\ &= c_1^2 + c_2^2. \end{aligned}$$

Therefore each trajectories are circles centered at the origin. See Figure 4.3, where four trajectories (circles) are displayed together with the direction field of the system. We can see that the trajectories are tangent to the direction field.

□

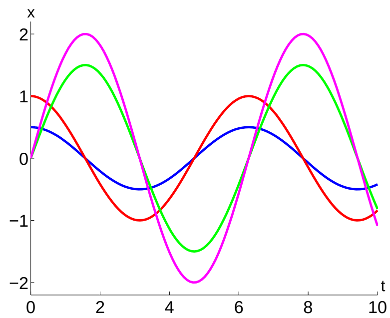

Figure 4.1: $x(t)$

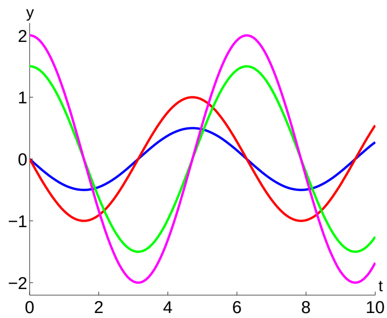

Figure 4.2: $y(t)$

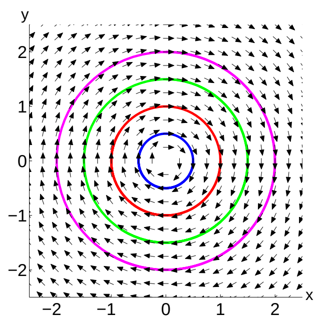

Figure 4.3: phase plain

The next results summarize some important geometrical properties of autonomous systems.

Theorem 4.3 *Different trajectories of an autonomous differential equation do not intersect each other.*

Theorem 4.4 *A trajectory of an autonomous differential equation is either*

(i) a simple curve (which has no intersection with itself) or

(ii) a simple closed curve or

(iii) a single point.

A trajectory is a single point if and only if it corresponds to a constant equilibrium. It is a simple closed curve if and only if it corresponds to a periodic solution.

Theorem 4.5 *If for a solution* $\mathbf{x}$ *the limit*

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{u} \qquad (or \lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{u})$$

exists and it is finite, then $\mathbf{u}$ *is an equilibrium point of Eq. (4.1).*

4.2 Stability notions

The norm of a vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is defined by $\|\mathbf{x}\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. Let $\mathbf{u}$ be a fixed equilibrium of Eq. (4.1). The equilibrium $\mathbf{u}$ is called *stable*, if for every $\varepsilon > 0$ there exists $\delta > 0$, such that if $\|\mathbf{z} - \mathbf{u}\| < \delta$, then $\|\mathbf{x}(t; \mathbf{z}) - \mathbf{u}\| < \varepsilon$ holds for all $t \geq 0$. The equilibrium $\mathbf{u}$ is called *unstable*, if it is not stable.

The equilibrium $\mathbf{u}$ is *asymptotically stable*, if it is stable, and there exists $\sigma > 0$ such that if $\|\mathbf{z} - \mathbf{u}\| < \sigma$, then $\lim_{t\to\infty} \mathbf{x}(t; \mathbf{z}) = \mathbf{u}$.

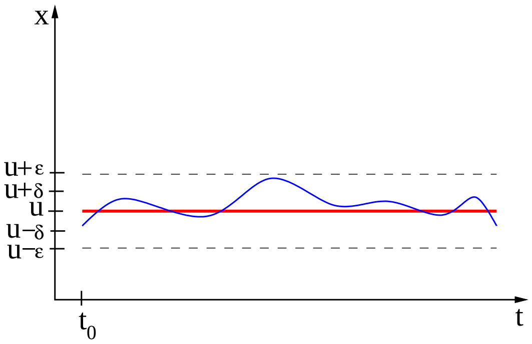

Figure 4.4: stable equilibrium

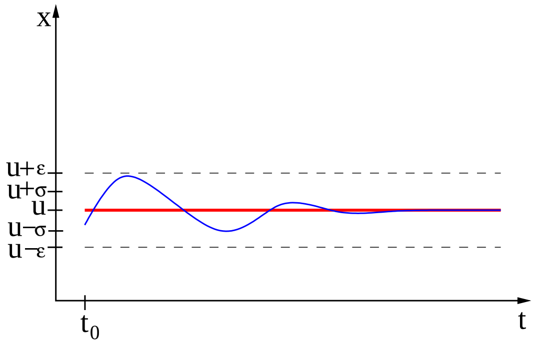

Figure 4.5: asymptotically stable equilibrium

If an equilibrium is stable, then all solutions remain close to it assuming that the initial condition is close enough to the equilibrium (see Figure 4.4). If the equilibrium is asymptotically stable, then all solutions converge to the equilibrium, assuming the IC is close enough to the equilibrium (see Figure 4.5).

The above definitions of stability are called *Liapunov stability*. We note that there are many different variants of the notion of stability investigated in the engineering and mathematical literature.

Example 4.6 Consider the scalar equation

$$x' = x - x^3.$$

It has three equilibriums: $u = 0, 1$ and -1, which are the roots of the equation $u - u^3 = 0$. In Figure 4.6, some integral curves and the directional field are displayed. It can be seen that if the solution is between the lines $x = 0$ and $x = 1$, then the right hand side of the equation is positive, so the derivative of the solution is positive, therefore the solution is increasing. It can be shown that all solutions in this horizontal strip converge to 1. Similar argument gives that all solutions which start above 1 will converge to 1 monotone decreasingly, all solutions which start between 0 and -1 converge to -1 monotone decreasingly, and finally, all solutions which start below -1 converge to -1 monotone increasingly. This means, in particular, that 1 and -1 are asymptotically stable equilibriums and 0 is an unstable equilibrium.

□

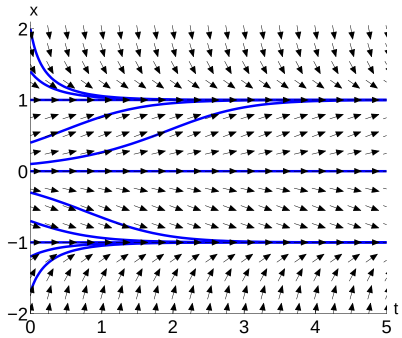

Figure 4.6: integral curves of Example 4.6

Example 4.7 Consider again the 2×2 system examined in Example 4.2 whose trajectories can be seen in Figure 4.3. The system has only one equilibrium, $\mathbf{u} = \mathbf{0}$. This is stable, since if we take any $\varepsilon > 0$, then all trajectories starting from the neighborhood of the origin of radius $\delta = \varepsilon$ will remain inside the neighborhood of radius ε, because all trajectories are circles around the origin. The equilibrium is not asymptotically stable since the trajectories do not converge to the origin. This can also be seen from the formula of the solutions $x(t) = c_1 \cos t + c_2 \sin t$ and $y(t) = -c_1 \sin t + c_2 \cos t$, because its amplitude is constant $\sqrt{x^2(t) + y^2(t)} = \sqrt{c_1^2 + c_2^2}$. □

4.3 Scalar nonlinear autonomous equations

Example 4.8 Consider again the equation $x' = x - x^3$ examined in Example 4.6. There is a simple graphical visualization of the trajectories of the scalar equation. Since it is a scalar equation, the trajectories are one-dimensional objects, so they are graphed on the real line.

The trajectory of an equilibrium is a point on the real line. The trajectory of a solution corresponding to an initial value between 0 and 1 is the open interval $(0, 1)$, since, as it was explained in Example 4.6, the solution approaches to 1 monotone increasingly as $t \to \infty$, and it can be shown that the solution tends to 0 monotone decreasingly as $t \to -\infty$. Hence the trajectory of any solution starting between 0 and 1 is the whole interval (0,1). Similarly, $(-1, 0)$, $(1, \infty)$ and $(-\infty, -1)$ are trajectories of the equation besides the three equilibrium points $-1, 0$ and 1.

This property can be easily visualized in Figure 4.7. Here the horizontal axis is the x-axis, the phase space of the solution. The equilibrium points are denoted by circles on the x-axis, and they divide the x-axis into 4 intervals.

On the vertical axis, the graph of the right-hand side of the equation, i.e., in this case the function $x - x^3$ is plotted. On the interval $(0, 1)$ the function is positive, therefore the solution is monotone increasing. Therefore the interval $(0, 1)$ on the x-axis is directed to the right. Similarly, the interval $(-\infty, -1)$ is directed to the right since the graph on this interval is above the x-axis. The intervals $(-1, 0)$ and $(1, \infty)$ are directed to the left, since the graph over these intervals are below the x-axis.

The equilibrium points -1 and 1 are asymptotically stable, since all solution starting close to the equilibrium tend to it. But 0 is an unstable equilibrium, since the solutions which start close to 0 go away from 0. In the figure, the stable equilibriums are denoted by a solid dot, the unstable equilibrium is denoted by an open circle. □

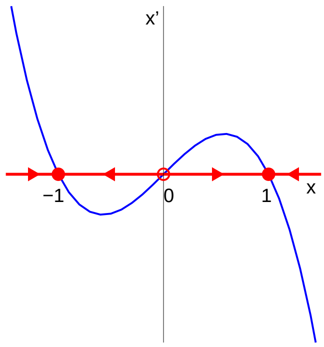

Figure 4.7: red: trajectories of $x' = x - x^3$, blue: graph of $x - x^3$

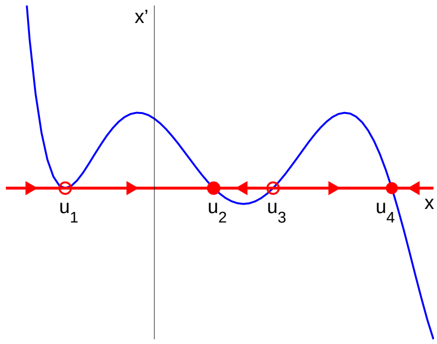

Figure 4.8: red: trajectories of $x' = f(x)$, blue: graph of $f(x)$

In general, consider the scalar nonlinear autonomous differential equation

$$x' = f(x). \tag{4.1}$$

The equilibriums of Eq. (4.1) are the solutions of $f(u) = 0$. The dynamics of the solutions can be visualized in Figure 4.8. In this figure, Eq. (4.1) has four equilibriums, u_1, u_2, u_3 and u_4. u_1 and u_3 are unstable, and u_2 and u_4 are asymptotically stable. The asymptotically stable and the unstable equilibriums are represented by a solid and open circles, respectively.

We can see that a trajectory of a scalar vector field can converge to an equilibrium in a monotonic way, or it can converge to ∞ or $-\infty$, or it is a single point (an equilibrium point). This is the only dynamics which can happen, no oscillation can occur in the scalar case. We will see that more complicated dynamics can happen in higher dimension.

4.4 Two-dimensional autonomous homogeneous linear systems

Consider

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \tag{4.1}$$

where $\mathbf{A}$ is a 2×2 invertible matrix. Then the only equilibrium of the equation is $\mathbf{u} = \mathbf{0}$. It is known from linear algebra that $\mathbf{A}$ is invertible if and only if 0 is not an eigenvalue of $\mathbf{A}$.

Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$. We consider 6 cases.

Case 1: $\lambda_1 \neq \lambda_2$ are real eigenvalues of $\mathbf{A}$ of the same sign $(\lambda_1\lambda_2 > 0)$

In this case the real eigenvalues are either both positive or both negative. The general solution of the equation is

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\lambda_1 t} \boldsymbol{\xi}^{(1)} + c_2 \mathrm{e}^{\lambda_2 t} \boldsymbol{\xi}^{(2)}, \tag{4.2}$$

where c_1, c_2 are arbitrary constants and $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}$ are the eigenvectors of $\mathbf{A}$.

Consider the special case when $c_2 = 0$ and $c_1 \neq 0$. Then the corresponding solution is $\mathbf{x} = c_1 \mathrm{e}^{\lambda_1 t} \boldsymbol{\xi}^{(1)}$, i.e.,

$$\begin{aligned} x_1 &= c_1 \mathrm{e}^{\lambda_1 t} \xi_1^{(1)} \\ x_2 &= c_1 \mathrm{e}^{\lambda_1 t} \xi_2^{(1)}. \end{aligned}$$

Dividing the two equations, we get

$$\frac{x_2}{x_1} = \frac{\xi_2^{(1)}}{\xi_1^{(1)}},$$

therefore

$$x_2 = \frac{\xi_2^{(1)}}{\xi_1^{(1)}} x_1.$$

Its graph is a line through the origin with slope $\xi_2^{(1)}/\xi_1^{(1)}$. Hence the trajectory is the open half-line through the vector $\boldsymbol{\xi}^{(1)}$ if $c_1 > 0$, and the open half-line through the vector $-\boldsymbol{\xi}^{(1)}$ if $c_1 < 0$. Similarly, the trajectory of the solution corresponding to $c_1 = 0$ and $c_2 \neq 0$ is the open half-line starting at the origin through the vector $\boldsymbol{\xi}^{(2)}$ or the opposite direction $-\boldsymbol{\xi}^{(2)}$.

In the general case when $c_1 \neq 0$ and $c_2 \neq 0$ the trajectories are curves defined by the system

$$x_1 = c_1 \mathrm{e}^{\lambda_1 t} \xi_1^{(1)} + c_2 \mathrm{e}^{\lambda_2 t} \xi_1^{(2)} \tag{4.3}$$

$$x_2 = c_1 \mathrm{e}^{\lambda_1 t} \xi_2^{(1)} + c_2 \mathrm{e}^{\lambda_2 t} \xi_2^{(2)}. \tag{4.4}$$

If both λ_1 and λ_2 are negative, then the above equations yield $x_1(t) \to 0$ and $x_2(t) \to 0$ if $t \to \infty$, hence the curves approach the origin as $t \to \infty$. Therefore in this case the origin is asymptotically stable, and all trajectories are directed towards the origin. In the opposite case when both eigenvalues are positive then $x_1(t) \to \infty$ and $x_2(t) \to \infty$ as $t \to \infty$. Hence the trajectories are directed away from the origin and go to infinity.

Suppose $\lambda_1 < \lambda_2 < 0$. Then Eq. (4.2) yields

$$\mathbf{x}(t) = \mathrm{e}^{\lambda_2 t} \left(c_1 \mathrm{e}^{-(\lambda_2 - \lambda_1)t} \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\xi}^{(2)} \right). \tag{4.5}$$

Since $\mathrm{e}^{-(\lambda_2-\lambda_1)t} \to 0$ as $t \to \infty$, for large t we have

$$\mathbf{x}(t) \approx c_2 \mathrm{e}^{\lambda_2 t} \boldsymbol{\xi}^{(2)}, \tag{4.6}$$

hence the graph of the solution is close to the line determined by the vector $\boldsymbol{\xi}^{(2)}$. It can be shown that these curves are tangent to the line through $\boldsymbol{\xi}^{(2)}$. A typical phase portrait of the system can be seen in Figure 4.9. The equilibrium is called an *asymptotically stable node* or sometimes an *asymptotically stable improper node*.

Conversely, if $0 < \lambda_2 < \lambda_1$, then the difference is that as $t \to \infty$ both coordinates of the solution go to infinity along the trajectory. And when $t \to -\infty$, then the solutions converge to the origin. It can be shown that if t is negative and its absolute values is large, then from (4.5) it follows (4.6), hence the trajectories converge to the line through $\boldsymbol{\xi}^{(2)}$ as $t \to -\infty$. Typical trajectories can be seen in Figure 4.10. In this case the the origin is called *unstable node* or an *unstable improper node*.

Case 2: $\lambda_1 = \lambda_2$ are real eigenvalues of A with geometric multiplicity 2

The solution of Eq. (4.1) is again (4.2), but this parametric equation defines a half-line not only for the case $c_1 = 0$ or $c_2 = 0$, but also for $c_1 \neq 0$ and $c_2 \neq 0$. Indeed, in this case the system (4.3)-(4.4) has the form

$$x_1 = \mathrm{e}^{\lambda_1 t}\left(c_1 \xi_1^{(1)} + c_2 \xi_1^{(2)}\right)$$

$$x_2 = \mathrm{e}^{\lambda_1 t}\left(c_1 \xi_2^{(1)} + c_2 \xi_2^{(2)}\right),$$

which yields $x_2 = m x_1$ with $m = (c_1\xi_2^{(1)} + c_2\xi_2^{(2)})/(c_1\xi_1^{(1)} + c_2\xi_1^{(2)})$. Therefore in this case, all non-equilibrium trajectories are open half-lines starting at the origin. If $\lambda_1 < 0$, then all

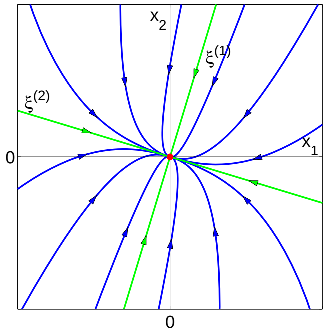

Figure 4.9: asymptotically stable node

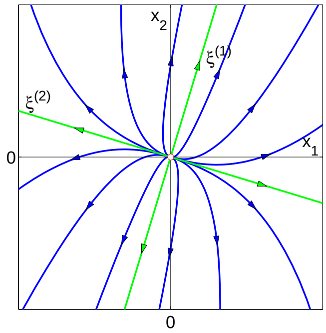

Figure 4.10: unstable node

trajectories are directed toward the origin, and in the case $\lambda_1 > 0$ all trajectories are directed away from the origin (see Figures 4.11 and 4.12). The equilibrium is called *node* or *proper node* (to distinguished it from Case 1), which is asymptotically stable if $\lambda_1 < 0$, and it is unstable if $\lambda_1 > 0$. In this case, the eigenvectors do not denote special direction. In fact, all nonzero vectors are eigenvectors.

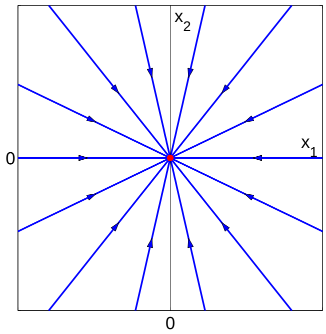

Figure 4.11: asymptotically stable node

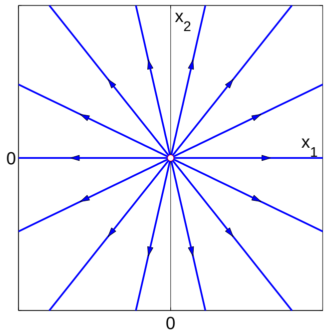

Figure 4.12: unstable node

Case 3: $\lambda_1 = \lambda_2$ real eigenvalues of A with geometric multiplicity 1

In this case, the general solution of (4.1) has the form

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\lambda_1 t} \boldsymbol{\xi}^{(1)} + c_2 \mathrm{e}^{\lambda_1 t} \Big(t \boldsymbol{\xi}^{(1)} + \boldsymbol{\eta} \Big),$$

where $\boldsymbol{\eta}$ is the generalized eigenvector corresponding to λ_1. For $c_2 = 0$ and $c_1 \neq 0$, we get the half-line trajectories from the origin through the vectors $\boldsymbol{\xi}^{(1)}$ and $-\boldsymbol{\xi}^{(1)}$, respectively. In general, the solution $\mathbf{x}$ is a multiple with the scalar time-dependent factor $\mathrm{e}^{\lambda_1 t}$ of the vector

$$\mathbf{v}(t) = c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\eta} + t c_2 \boldsymbol{\xi}^{(1)}.$$

The graph of $\mathbf{v}(t)$ is a line parallel to the vector $\boldsymbol{\xi}^{(1)}$. (See Figure 4.13, where it is the line through the vector $\mathbf{a} = c_1 \boldsymbol{\xi}^{(1)} + c_2 \boldsymbol{\eta}$ parallel to the vector $\boldsymbol{\xi}^{(1)}$.) Therefore all half-lines starting from the origin intersect this trajectory in a single point. The trajectory is plotted in Figure 4.13 which is tangent to the line through the origin with direction $\boldsymbol{\xi}^{(1)}$.

The equilibrium is also called *improper node* or sometimes *degenerate node*. It is asymptotically stable if $\lambda_1 < 0$, and it is unstable if $\lambda_1 > 0$ (see Figures 4.14 and 4.15).

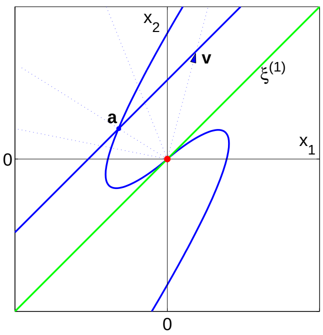

Figure 4.13: construction of the trajectory

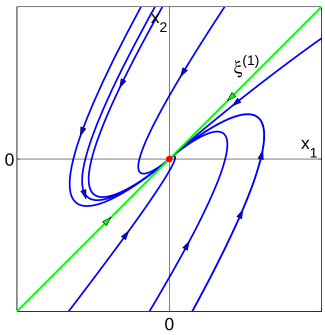

Figure 4.14: asymptotically stable improper node

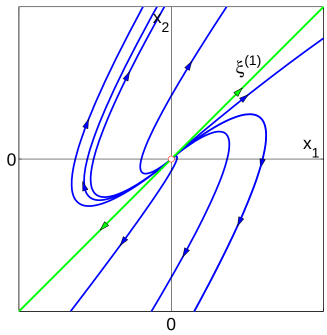

Figure 4.15: unstable improper node

Case 4: $\lambda_1 \neq \lambda_2$ real eigenvalues of A of opposite sign ($\lambda_1\lambda_2 < 0$)

Suppose, e.g., $\lambda_2 < 0 < \lambda_1$. The general solution of Eq. (4.1) is again (4.2), so there are two lines, more precisely, four half-line trajectories (the cases of $c_1 = 0$ or $c_2 = 0$). For large t $\mathrm{e}^{\lambda_2 t} \approx 0$, therefore

$$\mathbf{x}(t) \approx c_1 \mathrm{e}^{\lambda_1 t} \boldsymbol{\xi}^{(1)}$$

holds, so the trajectories approach the line determined by the vector $\boldsymbol{\xi}^{(1)}$. In the opposite case, when $t \to -\infty$ it holds $\mathrm{e}^{\lambda_1 t} \approx 0$, which yields that the trajectories approach the line determined by $\boldsymbol{\xi}^{(2)}$ as $t \to -\infty$. Therefore the shapes of the trajectories are similar to "hyperbolas", as it can be seen in Figure 4.16. The direction of the two half-lines corresponding to the positive eigenvalue is away from the origin, and the direction of the half-lines corresponding to negative eigenvalue is to the origin. The directions of the other trajectories are fit to these directions (see Figure 4.16). The line determined by the positive eigenvalue is called *unstable subspace*, the subspace corresponding to the negative eigenvalue is called *stable subspace*. All trajectories approach to the unstable subspace as $t \to \infty$, and to the stable subspace as $t \to -\infty$. The equilibrium is called *saddle point*. A saddle point is always unstable, since every solutions which do not start on the stable subspace go away from any neighborhood of the origin.

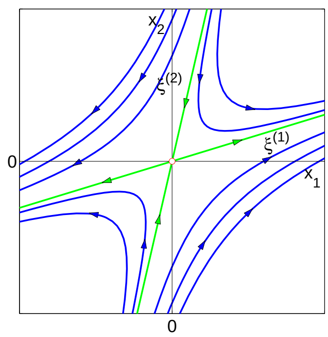

Figure 4.16: saddle point, unstable

Case 5: $\lambda_{1,2} = \alpha \pm i\beta$ ($\alpha \neq 0$) complex eigenvalues of A

Let $\boldsymbol{\xi} = \mathbf{u} + i\mathbf{v}$ be the eigenvector of $\mathbf{A}$ corresponding to $\lambda = \alpha + i\beta$. Then

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi} = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}) = \alpha\mathbf{u} - \beta\mathbf{v} + i(\beta\mathbf{u} + \alpha\mathbf{v}),$$

so taking the real and imaginary parts of the equation we get

$$\mathbf{A}\mathbf{u} = \alpha\mathbf{u} - \beta\mathbf{v} \qquad \text{and} \qquad \mathbf{A}\mathbf{v} = \beta\mathbf{u} + \alpha\mathbf{v}.$$

Let $\mathbf{T}$ be the 2×2 matrix whose first column is $\mathbf{u}$ and its second column is $\mathbf{v}$, i.e., $\mathbf{T} = (\mathbf{u}, \mathbf{v})$. Then

$$\mathbf{AT} = (\mathbf{Au}, \mathbf{Av}) = (\alpha\mathbf{u} - \beta\mathbf{v}, \beta\mathbf{u} + \alpha\mathbf{v}) = (\mathbf{u}, \mathbf{v}) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \mathbf{T} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

This implies that

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Let $\mathbf{x} = \mathbf{Ty}$, i.e., $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$. Then

$$\mathbf{y}' = \mathbf{T}^{-1}\mathbf{x}' = \mathbf{T}^{-1}\mathbf{Ax} = \mathbf{T}^{-1}\mathbf{ATy},$$

i.e., the new variable $\mathbf{y} = (y_1, y_2)^T$ satisfies the differential equation

$$\begin{aligned} y_1' &= \alpha y_1 + \beta y_2 \\ y_2' &= -\beta y_1 + \alpha y_2. \end{aligned} \tag{4.7}$$

Consider the polar coordinates form (r, θ) of (y_1, y_2). Then $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$ and thus

$$r^2 = y_1^2 + y_2^2 \qquad \text{and} \qquad \operatorname{tg}\theta = \frac{y_2}{y_1}.$$

Differentiating both sides of the first equation and using relations (4.7), we get

$$2rr' = 2y_1y_1' + 2y_2y_2' = 2y_1(\alpha y_1 + \beta y_2) + 2y_2(-\beta y_1 + \alpha y_2) = 2\alpha(y_1^2 + y_2^2) = 2\alpha r^2.$$

Therefore equation $r' = \alpha r$ holds along the motion, so $r = c_1 \mathrm{e}^{\alpha t}$. This yields that the distance from the origin along the motion increases exponentially if $\alpha > 0$, and it is exponentially decreasing to 0 if $\alpha < 0$.

Differentiating the equation of θ, we get

$$\frac{1}{\cos^2\theta}\theta' = \frac{y_2'y_1 - y_1'y_2}{y_1^2} = \frac{(-\beta y_1 + \alpha y_2)y_1 - (\alpha y_1 + \beta y_2)y_2}{y_1^2} = -\beta\frac{y_1^2 + y_2^2}{y_1^2} = -\beta\frac{r^2}{y_1^2},$$

so $\theta' = -\beta$, hence $\theta = -\beta t + \theta_0$. Therefore the point rotates with a constant speed around the origin along the motion. This yields that the trajectories of the y variable are spirals, see Figures 4.17 and 4.18.

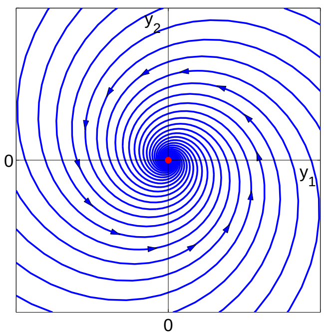

Figure 4.17: trajectories of Eq. (4.7), $\lambda = \alpha \pm i\beta$, $\alpha < 0$

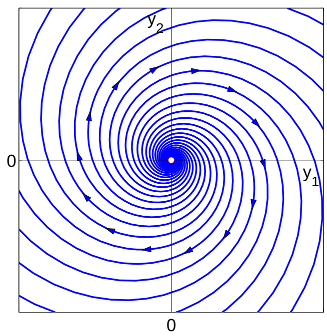

Figure 4.18: trajectories of Eq. (4.7), $\lambda = \alpha \pm i\beta$, $\alpha > 0$

Applying the linear transformation $\mathbf{T}$ on the spirals, we get curves of the form shown in Figures 4.19 and 4.20. The name of the equilibrium is *spiral point*, or *focus* in this case which is asymptotically stable if $\alpha < 0$, and it is unstable if $\alpha > 0$.

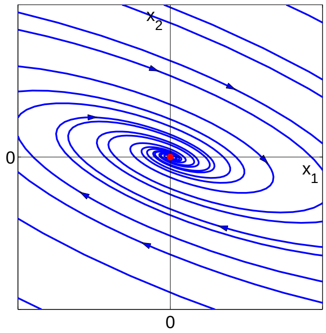

Figure 4.19: asymptotically stable spiral

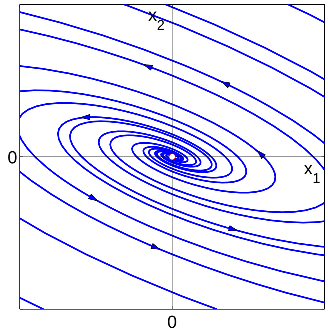

Figure 4.20: unstable spiral

Case 6: $\lambda_{1,2} = \pm i\beta$ are complex eigenvalues of A

The derivation of the differential equations for the polar coordinates r and θ are valid in the case when $\alpha = 0$, so in this case, we have $r' = 0$, hence r is constant along the solutions. Hence the trajectories of Eq. (4.7) are circles centered at the origin (see Figure 4.21). The linear transformation $\mathbf{T}$ deforms the circles into ellipses, see Figure 4.22. The equilibrium is called *center*. The center is always stable but it is not asymptotically stable.

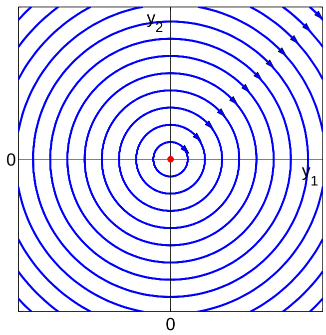

Figure 4.21: trajectories of Eq. (4.7), $\lambda = \pm i\beta$

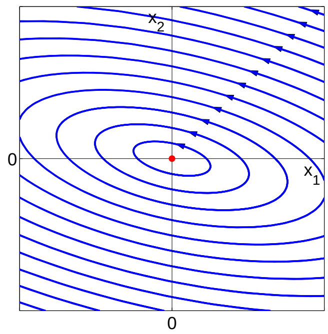

Figure 4.22: center, stable

4.5 Stability of linear systems

Consider the linear homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}, \tag{4.1}$$

where $A\colon [0,\infty) \to \mathbb{R}^{n\times n}$ is continuous. $\mathbf{u} = \mathbf{0}$ is an equilibrium of the system, i.e., the constant function $\mathbf{0}$ is a solution of the equation, and if we assume that the matrix $\mathbf{A}(t)$ is invertible for all $t \geq 0$, then $\mathbf{0}$ is the only equilibrium point.

The following theorem can be proved.

Theorem 4.9 *The equilibrium* $\mathbf{u} = \mathbf{0}$ *of Eq. (4.1) is*

(a) stable if and only if all solutions of Eq. (4.1) are bounded;

(b) asymptotically stable if and only if all solutions $\mathbf{x}$ *of Eq. (4.1) satisfy*

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}.$$

Consider now the autonomous version of Eq. (4.1), i.e., consider

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{4.2}$$

Then the stability of the trivial equilibrium $\mathbf{u} = \mathbf{0}$ is determined by the eigenvalues of the coefficient matrix $\mathbf{A}$.

Theorem 4.10 *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the $n \times n$ matrix $\mathbf{A}$. Then the equilibrium $\mathbf{u} = \mathbf{0}$ of Eq. (4.2) is*

(a) stable if and only if

$$\operatorname{Re} \lambda_i \leq 0, \qquad i = 1, \ldots, n,$$

and if $\operatorname{Re} \lambda_i = 0$ for some i, then the geometric multiplicity of λ_i equals to its algebraic multiplicity;

(b) asymptotically stable if and only if

$$\operatorname{Re} \lambda_i < 0, \qquad i = 1, \ldots, n.$$

In some special cases, the asymptotic stability can be determined without computing the eigenvalues of $\mathbf{A}$.

Theorem 4.11 *Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix which satisfies*

$$a_{ii} < 0, \qquad i = 1, \ldots, n.$$

If the matrix $\mathbf{A}$ is row diagonally dominant, i.e.,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n,$$

or if $\mathbf{A}$ is column diagonally dominant, i.e.,

$$|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|, \quad j = 1, \ldots, n,$$

then the equilibrium $\mathbf{u} = \mathbf{0}$ of Eq. (4.2) is asymptotically stable.

4.6 Stability of nonlinear systems

Consider first the so-called *quasi-linear differential equation*

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(\mathbf{x}), \tag{4.1}$$

where we assume $\mathbf{g}(\mathbf{0}) = \mathbf{0}$ and

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x}\|} = 0. \tag{4.2}$$

The first condition implies that $\mathbf{u} = \mathbf{0}$ is an equilibrium of Eq. (4.1), the second condition yields that the function $\mathbf{g}$ contains only terms which are higher-order than linear.

Omitting the nonlinear terms in Eq. (4.1), i.e., the function $\mathbf{g}$, we get the so-called *linearized equation*:

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{4.3}$$

The next theorem shows that the asymptotic stability and the instability properties of the equilibrium $\mathbf{u} = \mathbf{0}$ of the linearized equation Eq. (4.3) are preserved for the equilibrium $\mathbf{u} = \mathbf{0}$ of the nonlinear equation Eq. (4.1).

Theorem 4.12 *If the equilibrium* $\mathbf{u} = \mathbf{0}$ *of the linearized equation Eq. (4.3) is asymptotically stable, then the equilibrium* $\mathbf{u} = \mathbf{0}$ *of the quasi-linear equation Eq. (4.1) is also asymptotically stable.*

If the matrix coefficient $\mathbf{A}$ *of the linearized equation Eq. (4.3) has an eigenvalue with positive real part, then the equilibrium* $\mathbf{u} = \mathbf{0}$ *of the quasi-linear equation Eq. (4.1) is unstable.*

If the equilibrium $\mathbf{u} = \mathbf{0}$ of the linearized equation Eq. (4.3) is stable but not asymptotically stable, then the equilibrium $\mathbf{u} = \mathbf{0}$ of the quasi-linear equation can be asymptotically stable, stable or unstable depending on the nonlinear term $\mathbf{g}$ (see Examples 4.19 and 4.20 below).

In the two-dimensional case we can draw the trajectories of the quasi-linear equation in the phase plane. It can be shown that the shape of the trajectories are similar to the shape of the trajectories of the corresponding linearized equation, in a small neighborhood of the origin they can be obtained from the trajectories of the linearized equation using a nonlinear deformation.

Example 4.13 Determine the stability property of the trivial equilibrium $(0,0)$ of the nonlinear system

$$\begin{aligned} x' &= x + y - 2xy \\ y' &= 4x + y + x^2. \end{aligned}$$

Nonlinear terms of the system are

$$\mathbf{g}(x,y) = \begin{pmatrix} -2xy \\ x^2 \end{pmatrix}.$$

This function satisfies condition (4.2), since using the polar transformation $x = r\cos\theta$, $y = r\sin\theta$ we get

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{0}} \frac{\|\mathbf{g}(\mathbf{x})\|^2}{\|\mathbf{x}\|^2} &= \lim_{(x,y)\to(0,0)} \frac{(-2xy)^2 + x^4}{x^2+y^2} \\ &= \lim_{r\to 0} \frac{4r^4\cos^2\theta\sin^2\theta + r^4\cos^4\theta}{r^2} \\ &= \lim_{r\to 0} r^2(4\cos^2\theta\sin^2\theta + \cos^4\theta) = 0. \end{aligned}$$

The associated linearized equation is

$$\begin{aligned} x' &= x + y \\ y' &= 4x + y. \end{aligned}$$

The eigenvalues of the coefficient matrix

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

are 3 and -1, so the trivial equilibrium of the linearized equation is a saddle point, which is unstable. Since the coefficient matrix has a positive eigenvalue, we get that the trivial equilibrium $(0,0)$ of the nonlinear equation is also unstable. □

Consider now the general autonomous nonlinear system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \tag{4.4}$$

and let $\mathbf{u}$ be a fixed equilibrium. We are interested in examining the behavior of the difference $\mathbf{x} - \mathbf{u}$ for a solution $\mathbf{x}$ starting from an initial condition close to $\mathbf{u}$. So introduce the new variable

$$\mathbf{y} = \mathbf{x} - \mathbf{u}.$$

Then, using that $\mathbf{u}$ is constant, we get that $\mathbf{y}$ satisfies the differential equation

$$\mathbf{y}' = \mathbf{x}' = \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{y} + \mathbf{u}).$$

Consider the first-order Taylor approximation of $\mathbf{f}$ around $\mathbf{u}$, where the error term is denoted by $\mathbf{g}$:

$$\mathbf{f}(\mathbf{y} + \mathbf{u}) = \mathbf{f}(\mathbf{u}) + \mathbf{f}'(\mathbf{u})\mathbf{y} + \mathbf{g}(\mathbf{y})$$

Here $\mathbf{f}'$ denotes the Jacobian of $\mathbf{f}$, i.e., the $n \times n$ matrix where element in the ith row and the jth column is the partial derivative of the ith component function of $\mathbf{f}$ with respect to the jth variable.

Since $\mathbf{u}$ is an equilibrium, $\mathbf{f}(\mathbf{u}) = \mathbf{0}$, thus we get the quasi-linear system

$$\mathbf{y}' = \mathbf{f}'(\mathbf{u})\mathbf{y} + \mathbf{g}(\mathbf{y}).$$

Theorem 4.12 yields the following result.

Theorem 4.14 *Let $\mathbf{u}$ be an equilibrium of Eq. (4.4). Then if the trivial equilibrium $\mathbf{0}$ of the linearized equation*

$$\mathbf{y}' = \mathbf{f}'(\mathbf{u})\mathbf{y}$$

is asymptotically stable, then the equilibrium $\mathbf{u}$ of the nonlinear equation Eq. (4.4) is also asymptotically stable. If the matrix $\mathbf{f}'(\mathbf{u})$ has an eigenvalue with positive real part, then the equilibrium $\mathbf{u}$ of the nonlinear equation Eq. (4.4) is unstable.

Example 4.15 Find all equilibriums of the nonlinear system

$$\begin{aligned} x' &= x(3 - 2x - y) \\ y' &= y(2 - x - y), \end{aligned}$$

and determine the stability properties of the equilibriums.

The equilibriums are the solutions of the nonlinear algebraic system

$$\begin{aligned} x(3 - 2x - y) &= 0 \\ y(2 - x - y) &= 0. \end{aligned}$$

We have four cases:
(i) $x = 0$ and $y = 0$, hence $(0, 0)$ is an equilibrium.
(ii) $x = 0$ and $2 - x - y = 0$. Then the solution is $(0, 2)$.
(iii) $3 - 2x - y = 0$ and $y = 0$. The corresponding equilibrium is $(1.5, 0)$.
(iv) $3 - 2x - y$ and $2 - x - y = 0$. This yields the equilibrium $(1, 1)$.
The right-hand-side in vector form is

$$\mathbf{f}(x, y) = \begin{pmatrix} 3x - 2x^2 - xy \\ 2y - xy - y^2 \end{pmatrix},$$

its Jacobian is

$$\mathbf{f}'(x, y) = \begin{pmatrix} 3 - 4x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}.$$

Consider the four equilibriums:
(i) $(0, 0)$. In this case

$$\mathbf{f}'(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

which is diagonal matrix, so its eigenvalues are 3 and 2. Therefore the linearized equation is an unstable node, so the equilibrium $(0, 0)$ of the nonlinear system is also unstable.
(ii) $(0, 2)$. We have

$$\mathbf{f}'(0, 2) = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix},$$

which is a lower triangular matrix, so its eigenvalues are the diagonal elements, 1 and -2. Therefore, the trivial equilibrium of the linearized equation is a saddle point, which is unstable. Therefore the equilibrium $(0,2)$ of the nonlinear system is also unstable.

(iii) $(1.5, 0)$. In this case

$$\mathbf{f}'(1.5,0) = \begin{pmatrix} -3 & -1.5 \\ 0 & 0.5 \end{pmatrix},$$

which is upper triangular, so its eigenvalues are -3 and 0.5. Therefore the trivial equilibrium of the linearized equation is a saddle point, which is unstable. Therefore the equilibrium $(1.5,0)$ of the nonlinear system is also unstable.

(iv) $(1,1)$. We have

$$\mathbf{f}'(1,1) = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix},$$

which have eigenvalues $\lambda_{1,2} = \frac{-3\pm\sqrt{5}}{2}$. Therefore the trivial equilibrium of the linearized equation is an asymptotically stable node, so the equilibrium $(1,1)$ of the nonlinear equation is also asymptotically stable.

In Figure 4.23 we plotted the trajectories of the nonlinear system. We can see that in a small neighborhood of the given equilibrium the shape of the trajectories are similar to the shape of the trajectories of the associated linearized equation. □

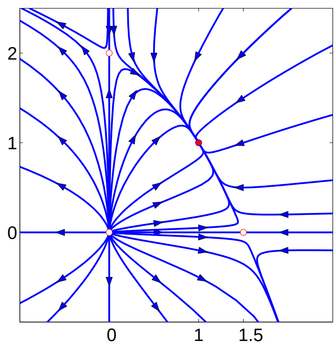

Figure 4.23: trajectories of the nonlinear system of Example 4.15

4.7 Liapunov functions

Let $U \subset \mathbb{R}^n$ be an open set satisfying $\mathbf{0} \in U$. A function $V\colon U \to \mathbb{R}$ is called *positive (negative) definite* if

$$V(\mathbf{0}) = 0 \qquad \text{and} \qquad V(\mathbf{x}) > 0 \quad (V(\mathbf{x}) < 0) \quad \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x} \in U.$$

A function V is called *positive (negative) semidefinite* if

$$V(\mathbf{0}) = 0 \qquad \text{and} \qquad V(\mathbf{x}) \geq 0 \quad (V(\mathbf{x}) \leq 0) \quad \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x} \in U.$$

Clearly, V is a negative (semi)definite function if and only if $-V$ is a positive (semi)definite function. In many applications, the domain of V is $U = \mathbb{R}^n$.

Example 4.16 Let

$$V(x,y) = ax^2 + bxy + cy^2$$

be a quadratic two-variable function. Suppose $a \neq 0$. We rewrite V as

$$V(x,y) = a\Big(x^2 + \frac{b}{a}xy\Big) + cy^2 = a\Big(\Big(x + \frac{b}{2a}y\Big)^2 - \frac{b^2}{4a^2}y^2\Big) + cy^2 = a\Big(x + \frac{b}{2a}y\Big)^2 + \frac{4ac-b^2}{4a}y^2.$$

This shows that if

$$a > 0 \qquad \text{and} \qquad 4ac - b^2 \geq 0,$$

then V is positive semidefinite. And if

$$a < 0 \qquad \text{and} \qquad 4ac - b^2 \geq 0,$$

then V is negative semidefinite.

V is positive definite if and only if

$$a > 0 \qquad \text{and} \qquad 4ac - b^2 > 0,$$

since if $V(x,y) = 0$, then

$$a\Big(x + \frac{b}{2a}y\Big)^2 = 0 \qquad \text{and} \qquad \frac{4ac-b^2}{4a}y^2 = 0,$$

which yields $y = 0$, and so $x = 0$.

Similarly, V is negative definite if and only if

$$a < 0 \qquad \text{and} \qquad 4ac - b^2 > 0.$$

□

The n-dimensional quadratic function has the general form

$$V\colon \mathbb{R}^n \to \mathbb{R}, \qquad V(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j.$$

We can rewrite it using a vector notation as

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x},$$

where

$$\mathbf{x} = (x_1,\ldots,x_n)^T, \qquad \mathbf{A} = (a_{ij}).$$

The following theorem is valid.

Theorem 4.17 (Sylvester) *The quadratic function $V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is positive definite if and only if all the* leading principal minors *of the matrix $\mathbf{A}$ are positive, i.e.,*

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \quad \ldots, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} > 0.$$

Consider again the nonlinear system

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}) \tag{4.1}$$

where we assume $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, i.e., $\mathbf{u} = \mathbf{0}$ is an equilibrium of the system. In this section, we investigate the stability of the trivial equilibrium $\mathbf{0}$.

The component functions of $\mathbf{f}$ are denoted by $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x}))^T$. Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be continuously partially differentiable with respect to all variables. Compute the derivative of the composite function $V(\mathbf{x}(t))$, where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$ is a solution of Eq. (4.1):

$$\begin{aligned}\frac{dV}{dt}(\mathbf{x}(t)) &= \frac{\partial V}{\partial x_1}(\mathbf{x}(t))x_1'(t) + \cdots + \frac{\partial V}{\partial x_n}(\mathbf{x}(t))x_n'(t) \\ &= \frac{\partial V}{\partial x_1}(\mathbf{x}(t))f_1(\mathbf{x}(t)) + \cdots + \frac{\partial V}{\partial x_n}(\mathbf{x}(t))f_n(\mathbf{x}(t)).\end{aligned}$$

The above formula motivates the following definition. The *derivative* of $\mathbf{V}$ *with respect to* Eq. (4.1) is defined by the formula

$$V'(\mathbf{x}) = \frac{\partial V}{\partial x_1}(\mathbf{x})f_1(\mathbf{x}) + \cdots + \frac{\partial V}{\partial x_n}(\mathbf{x})f_n(\mathbf{x}).$$

If the function $V'(\mathbf{x})$ is negative definite, then for every solution $\mathbf{x}(t)$ the composite function $V(\mathbf{x}(t))$ is monotone decreasing. Moreover, if $V(\mathbf{x}(t)) \to 0$ as $t \to \infty$, then $\mathbf{x}(t) \to \mathbf{0}$ holds. This is the background of the following theorem.

Theorem 4.18 (Liapunov) *Suppose* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $U \subset \mathbb{R}^n$ *is an open set,* $\mathbf{0} \in U$. *Let* $V\colon U \to \mathbb{R}$ *be continuously differentiable with respect to all variables.*

(a) If V is positive definite and V' is negative semidefinite, then the equilibrium $\mathbf{0}$ of Eq. (4.1) is stable.

(b) If V is positive definite and V' is negative definite, then the equilibrium $\mathbf{0}$ of Eq. (4.1) is asymptotically stable.

(c) If in every neighborhood of $\mathbf{0}$ there exists $\mathbf{x}$ such that $V(\mathbf{x}) > 0$ and V' is positive definite, then the equilibrium $\mathbf{0}$ of Eq. (4.1) is unstable.

A function V is called *Liapunov function* if its is positive definite and its derivative V' with respect to Eq. (4.1) is negative semidefinite.

Example 4.19 Finding a Liapunov function in the form $V(x, y) = ax^2 + by^2$ show that the trivial equilibrium $(0, 0)$ of the system

$$\begin{aligned} x' &= -x^3 - 2xy \\ y' &= 2x^2 - 6y \end{aligned}$$

is asymptotically stable.

Since we need V to be positive definite, we assume $a > 0$ and $b > 0$. Compute

$$V'(x, y) = 2ax(-x^3 - 2xy) + 2by(2x^2 - 6y) = -2ax^4 + (4b - 4a)x^2y - 12by^2.$$

For example if $a = b$, e.g., $a = b = 1$, then the term x^2y cancels out, so

$$V'(x, y) = -2x^4 - 12y^2$$

is negative definite. Therefore Theorem 4.18 yields that the trivial equilibrium is asymptotically stable.

We note that the method of linearization does not work for this system, because its linearized systems is $x' = 0$, $y' = 0$, which is stable but not asymptotically stable. Therefore Theorem 4.14 is not applicable. □

Example 4.20 Finding a Liapunov function in the form $V(x,y) = ax^2 + by^2$ show that the trivial equilibrium of the system

$$\begin{aligned} x' &= -x + 5y^2 \\ y' &= -3xy \end{aligned}$$

is stable.

Assuming $a > 0$ and $b > 0$ yields that V is positive definite. Consider

$$V'(x,y) = 2ax(-x + 5y^2) + 2by(-3xy) = -2ax^2 + (10a - 6b)xy^2.$$

We can see that if $10a - 6b \neq 0$, then the term $(10a - 6b)x^2y$ may take positive value, so the sign of V' may be positive. Therefore V' is not necessary negative semidefinite. Therefore, we select the parameters so that $10a - 6b = 0$ holds. For example, let $a = 3$ and $b = 5$. Then $V'(x,y) = -6x^2$, which is negative semidefinite, but not negative definite, since $V(0,y) = 0$ for all y. So we can conclude that the trivial equilibrium is stable. □

4.8 Applications

Example 4.21 (damped pendulum) Consider again the equation of the pendulum which was investigated in Example 2.26:

$$\theta'' + \frac{\gamma}{m}\theta' + \frac{g}{L}\sin\theta = 0. \tag{4.1}$$

Here we suppose that $\gamma > 0$, i.e., the damping force cannot be neglected. Using the new variables $x_1 = \theta$ and $x_2 = \theta'$, we get an equivalent system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{g}{L}\sin x_1 - \frac{\gamma}{m}x_2. \end{aligned}$$

Its equilibriums are solutions of the algebraic system

$$\begin{aligned} x_2 &= 0 \\ -\frac{g}{L}\sin x_1 - \frac{\gamma}{m}x_2 &= 0, \end{aligned}$$

which are $x_1 = k\pi$ $(k \in \mathbb{Z})$ and $x_2 = 0$. There are infinitely many equilibriums, but they determine two positions of the motions: the lower (l is even) and upper (l is odd) positions of the pendulum, where it is in rest.

Use the method of linearization to determine the stability of the equilibrium points. Compute first the Jacobian of the function

$$F(x_1,x_2) = \begin{pmatrix} x_2 \\ -\frac{g}{L}\sin x_1 - \frac{\gamma}{m}x_2 \end{pmatrix}.$$

We have

$$F'(x_1,x_2) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 & -\frac{\gamma}{m} \end{pmatrix}. \tag{4.2}$$

Consider two cases:

1. Suppose $k = 2j$ is even (*lower rest position*). Then

$$F'(2j\pi,0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{\gamma}{m} \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_{1,2} = \frac{-\frac{\gamma}{m} \pm \sqrt{\frac{\gamma^2}{m^2} - \frac{4g}{L}}}{2}. \tag{4.3}$$

We distinguish three subcases:

(a) $0 < \frac{\gamma^2}{m^2} < \frac{4g}{L}$ (small damping). Then the eigenvalues are complex numbers with negative real parts. Therefore, the trivial equilibrium is an asymptotically stable spiral, and hence the equilibrium of the nonlinear system is also asymptotically stable.

(b) $\frac{\gamma^2}{m^2} = \frac{4g}{L}$ (critical damping). In this case, the eigenvalue of the matrix is a negative real number which has algebraic multiplicity 2 and geometric multiplicity 1. Therefore the trivial equilibrium of the linearized equation is an asymptotically stable improper node, and hence the equilibrium of the nonlinear system is also asymptotically stable.

(c) $\frac{\gamma^2}{m^2} > \frac{4g}{L}$ (large damping). In this case, the two eigenvalues are both real, and we can check that both are negative. Therefore the trivial equilibrium of the linearized equation is an asymptotically stable node, and hence the equilibrium of the nonlinear system is also asymptotically stable.

We have seen that in all the above cases the lower rest position is asymptotically stable.

2. Let $k = 2j+1$ be odd (*upper rest position*). Then

$$F'((2j+1)\pi, 0) = \begin{pmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{\gamma}{m} \end{pmatrix},$$

therefore its eigenvalues are

$$\lambda_{1,2} = \frac{-\frac{\gamma}{m} \pm \sqrt{\frac{\gamma^2}{m^2} + \frac{4g}{L}}}{2}.$$

These are always real numbers, one is positive and the other is negative. Therefore the trivial equilibrium of the linearized equation is a saddle point, which is unstable, and hence the equilibrium of the nonlinear system is also unstable.

Some numerically generated trajectories are plotted in Figure 4.24 in the case of a small damping. We can see that the shapes of the trajectories of the linearized system are preserved in a small neighborhood of the equilibrium point for the nonlinear system. The trajectories are deformations of a spiral in the lower rest positions and deformations of a saddle point in the upper rest positions. □

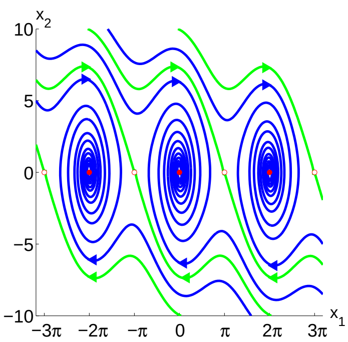

Figure 4.24: $m = 1$, $L = 1$, $\gamma = 0.5$

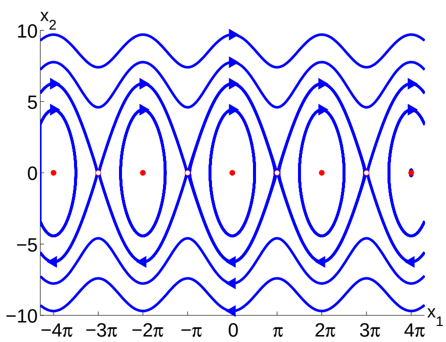

Figure 4.25: $m = 1$, $L = 1$, $\gamma = 0$

Example 4.22 (undamped pendulum) Consider again the pendulum equation Eq. (4.1) in the case when the damping is omitted, i.e., $\gamma = 0$:

$$\theta'' + \frac{g}{L}\sin\theta = 0. \tag{4.4}$$

Rewriting it in a system form, we get

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{g}{L}\sin x_1. \end{aligned}$$

Here, similar to Example 4.21, the equilibrium points are $(k\pi, 0)$ where $k \in \mathbb{Z}$.

In the upper rest position (when k is odd) the Jacobian matrix (4.2) with $\gamma = 0$ has one positive and one negative eigenvalue, so the equilibrium of the nonlinear system is unstable by the method of linearization.

In the lower rest points (for even k), it follows from relation (4.3) that the Jacobian has purely imaginary eigenvalues. Therefore the trivial equilibrium of the linearized equation is a center, so it is stable but not asymptotically stable. Therefore the method of linearized stability does not work for this case.

We apply the method of Liapunov functions in this case. In conservative mechanical systems the total energy is constant during the motion. In this case, we can select the total energy function as the Liapunov function. In the pendulum system, the potential energy of the mass m, i.e., the work done in lifting the mass from the lowest position to the position with angle $\theta = x_1$ is $mgL(1-\cos x_1)$. Its kinetic energy is $\frac{1}{2}mL^2(\theta')^2 = \frac{1}{2}mL^2x_2^2$. Consider therefore the function

$$V(x_1, x_2) = mgL(1-\cos x_1) + \frac{1}{2}mL^2x_2^2,$$

which is the total energy function. Let U be an open neighborhood of $(0, 0)$ which does not contain other equilibrium points. Then $1 - \cos x_1 > 0$ holds for $x_1 \neq 0$ in U, therefore V is positive definite on U. Compute V':

$$\begin{aligned} V'(x_1, x_2) &= mgL\sin x_1 \cdot x_1' + mL^2x_2x_2' \\ &= mgL\sin x_1 \cdot x_2 - mL^2x_2\frac{g}{L}\sin x_1 \\ &= 0. \end{aligned}$$

In particular, we got that V' is negative semidefinite. Therefore Theorem 4.18 yields that the origin is stable.

Consider now the general $(2j\pi, 0)$ lower rest point. Introduce the new variables $y_1 = x_1 - 2j\pi$ and $y_2 = x_2$. Then $y_1' = x_1' = x_2 = y_2$, $y_2' = x_2' = -\frac{g}{L}\sin x_1 = -\frac{g}{L}\sin(y_1 + 2j\pi) = -\frac{g}{L}\sin y_1$, hence we get

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{g}{L}\sin y_1. \end{aligned}$$

We have seen that the origin is a stable equilibrium of this system, it follows that $(2j\pi, 0)$ is a stable equilibrium of the pendulum system.

In Figure 4.25, some trajectories of the undamped pendulum system can be seen. The lower rest positions are deformations of a center, and the upper rest points are deformations of a saddle points. We can observe that there are special trajectories which connect the consecutive upper rest positions, i.e., there is a special starting position and initial velocity that the motion approaches to the upper rest position as $t \to \infty$. If the initial velocity is smaller then the pendulum oscillates periodically (its trajectory is a closed curve). And if the initial velocity is larger than this critical velocity, then the pendulum will rotate around forever. $\square$

Chapter 5
Elements of bifurcation in differential equations

In this chapter, we present several simple examples which illustrate some basic notions of bifurcation theory.

5.1 Scalar differential equations

We have seen in Section 4.3 that the dynamics of a scalar differential equation is simple. The main question is to determine the equilibriums and their stability properties. The situation is more interesting if the equation contains one or more parameters, and we are interested in how the dynamics changes when we change the parameters. The goal is to find a critical parameter value or values where passing this critical value some qualitative property of the differential equation (e.g., the number of equilibrium points or their stability property) changes. Such critical parameter value is called *bifurcation point*, change in dynamics is called *bifurcation*. In this section without trying to present a complete theory, we rather show some typical elementary examples and notions.

Example 5.1 Consider the scalar autonomous differential equation

$$x' = x^2 + a, \tag{5.1}$$

where $a \in \mathbb{R}$ is a parameter in the equation.

First compute the equilibriums of Eq. (5.1), i.e., solve the algebraic equation $u^2 + a = 0$. Depending on the parameter a is positive, 0 or negative, this equations has 2, 1 or 0 solutions, respectively, see Figures 5.1–5.3. Here $a = 0$ is a critical value, since if the parameter value is below it, the equation has 2 equilibriums, and above this critical parameter value the equation has no equilibrium. We can see that as a approaches to 0 from below then the two equilibriums, $u_1 = -\sqrt{-a}$ and $u_2 = \sqrt{-a}$ both approach to 0, to the only equilibrium point in the case when $a = 0$. Further, if $a > 0$ but arbitrary small, then the equilibrium point disappears. Such a situation where equilibriums are destroyed or created as the parameter approaches to the critical value is called *saddle-node bifurcation* or *fold bifurcation*.

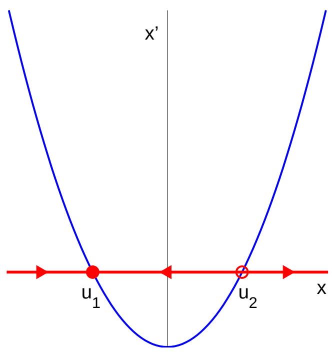

Figure 5.1: $a < 0$

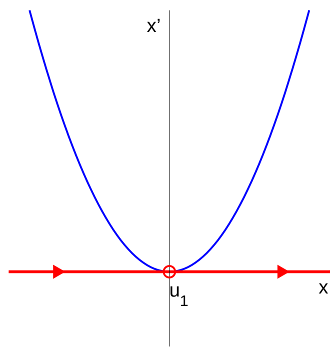

Figure 5.2: $a = 0$

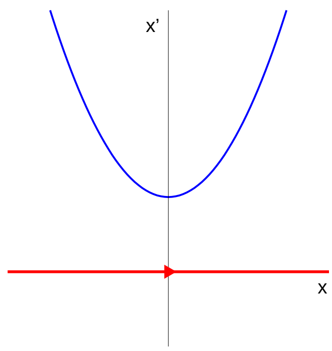

Figure 5.3: $a > 0$

The most common way to depict the bifurcation can be seen in Figure 5.4, where values of the equilibrium points are plotted as a function of the parameter. Such a picture is called *bifurcation diagram*. For $a < 0$ there are two equilibrium points, $x = \pm\sqrt{-a}$, which gives two branches of the parabola. The negative equilibrium is asymptotically stable, and the positive equilibrium is unstable. It is a common notation in bifurcation diagrams that the solid curve represents asymptotically stable equilibriums, the dotted curve denotes unstable equilibriums. In this picture, as the parameter increases and passes through the critical value, two equilibriums collide and disappear. For the equation $x' = x^2 - a$ the situation is opposite, as the parameter increases through the critical 0 parameter value, at 0 an equilibrium appears and for positive parameter values two equilibrium appear, see Figure 5.5. Both figures (and similar figures) describe saddle-node bifurcation. □

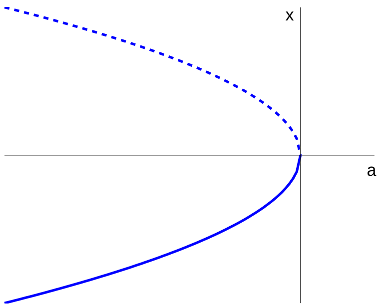

Figure 5.4: saddle-node, $x' = x^2 + a$

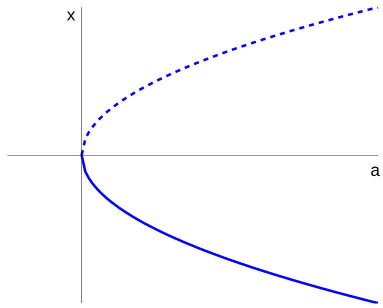

Figure 5.5: saddle-node, $x' = x^2 - a$

Example 5.2 Consider the scalar equation

$$x' = ax - x^2, \tag{5.2}$$

where $a \in \mathbb{R}$ is a parameter. It is easy to check that for $a \neq 0$ the equation has two equilibriums, $u_1 = 0$ and $u_2 = a$. For $a = 0$ the two equilibriums coincide. For $a < 0$ equilibrium u_1 is asymptotically stable, the equilibrium u_2 is unstable, but for $a > 0$ the situation is opposite, u_1 is unstable and u_2 is asymptotically stable. See the bifurcation diagram in Figure 5.9. Such a situation where stability properties of two branches of equilibrium points interchange is called *transcritical bifurcation.* □

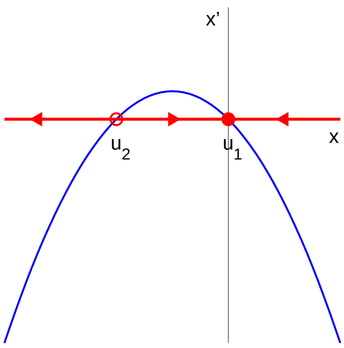

Figure 5.6: $a < 0$

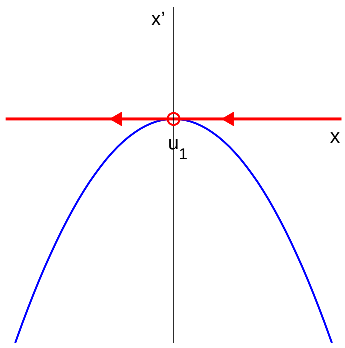

Figure 5.7: $a = 0$

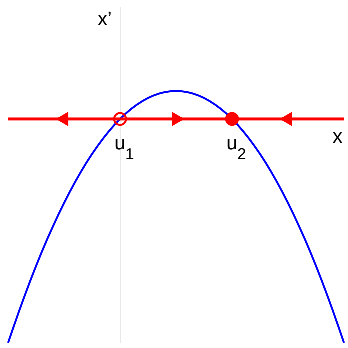

Figure 5.8: $a > 0$

Example 5.3 Consider the scalar equation

$$x' = ax - x^3, \tag{5.3}$$

where $a \in \mathbb{R}$ is a parameter. For $a < 0$ the equation has only one equilibrium, $u_1 = 0$, for $a > 0$ it has three equilibriums, $u_1 = 0$, $u_2 = -\sqrt{a}$ and $u_3 = \sqrt{a}$. At $a = 0$ all equilibriums coincide. It follows from Figure 5.10 and 5.11 that u_1 is asymptotically stable for $a \leq 0$, and it becomes unstable for $a > 0$ (see Figure 5.12). For $a > 0$, the other two equilibriums are

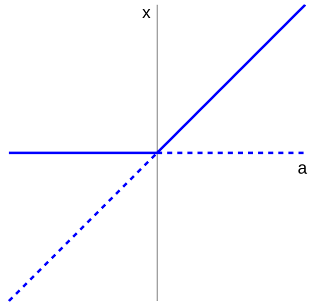

Figure 5.9: transcritical bifurcation, $x' = ax - x^2$

asymptotically stable. The bifurcation diagram can be seen in Figure 5.13. Such bifurcation where two equilibriums appear or disappear is called *pitchfork bifurcation.* □

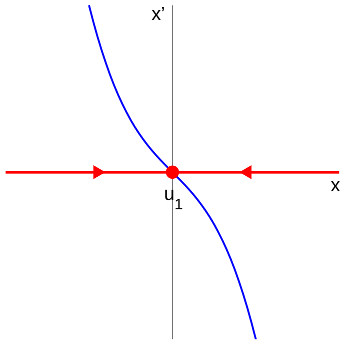

Figure 5.10: $a < 0$

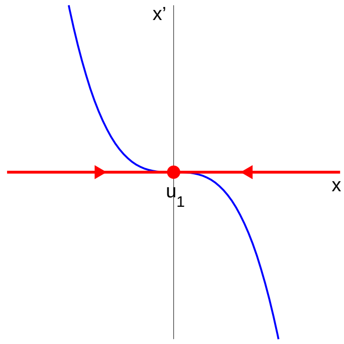

Figure 5.11: $a = 0$

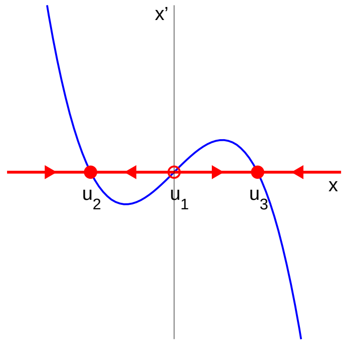

Figure 5.12: $a > 0$

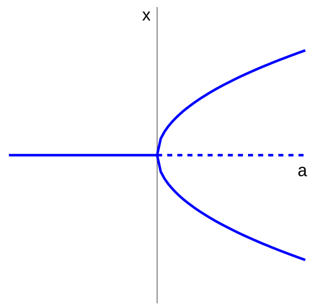

Figure 5.13: pitchfork bifurcation, $x' = ax - x^3$

Example 5.4 Assume that we have a population which is described by the logistic equation (see Example 1.21)

$$P' = rP\left(1 - \frac{P}{K}\right).$$

We assume that there is a continuous harvesting (fishing in the lake, hunting animals, or death due to predators) of the population which has the form $\frac{AP}{B+P}$, where A ad B are positive constants, i.e., the equation describing the change of the population is

$$P' = rP\left(1 - \frac{P}{K}\right) - \frac{AP}{B+P}.$$

The specific form of harvesting means that the rate tends monotone increasingly to a constant A as $P \to \infty$. Using the new variable $x(t) = P(t/r)/K$ we can transform the above equation into

$$x' = x(1-x) - \frac{ax}{b+x}, \tag{5.4}$$

where $a = A/(rK)$ and $b = B/K$ are positive parameters.

We can rewrite Eq. (5.4) in the form $x' = f(x)$, where

$$f(x) = x(g_1(x) - g_2(x)), \qquad g_1(x) = 1 - x \qquad \text{and} \qquad g_2(x) = \frac{a}{b+x}.$$

Clearly, u is an equilibrium of Eq. (5.4) if either $u = 0$ or $g_1(u) = g_2(u)$. Therefore $u_0 = 0$ is always an equilibrium of Eq. (5.4).

We fix $b = 1$ first, and investigate the dependence of Eq. (5.4) on the single parameter a. If $a < 1$, then the graphs of g_1 and g_2 have two intersections $u_1 = -\sqrt{1-a}$ and $u_2 = \sqrt{1-a}$ (see Figure 5.14). It is easy to check that f is positive on $(-1, u_1)$ and $(0, u_2)$, and it is negative on $(u_1, 0)$ and (u_1, ∞). This means that u_1 and u_2 are asymptotically stable and u_0 is unstable.

If $a \geq 1$ then the only equilibrium of Eq. (5.4) is $u_0 = 0$ (see Figures 5.15 and 5.16), which is asymptotically stable. This means that $a = 1$ is a pitchfork bifurcation for the case when $b = 1$ is fixed. The corresponding bifurcation diagram can be seen in Figure 5.17.

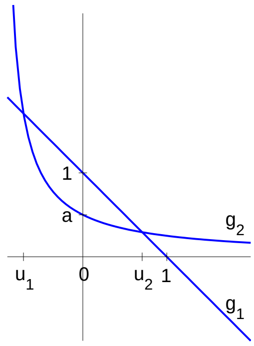

Figure 5.14: $a < 1$, $b = 1$

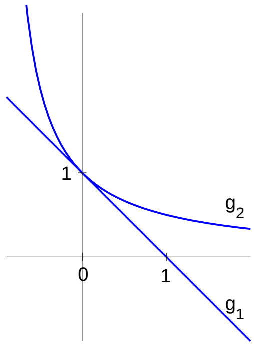

Figure 5.15: $a = 1$, $b = 1$

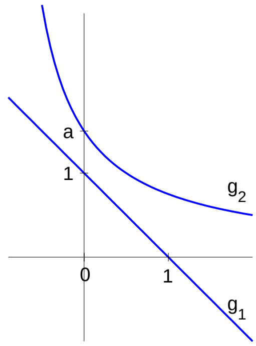

Figure 5.16: $a > 1$, $b = 1$

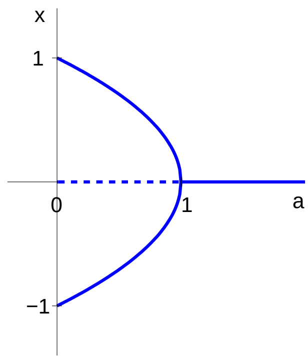

Figure 5.17: pitchfork bifurcation, $b = 1$

Now consider the case when $0 < b < 1$ is fixed. Then for $a < b$, we have three equilibriums again,

$$u_0 = 0, \qquad u_1 = \frac{1 - b - \sqrt{(1-b)^2 - 4(a-b)}}{2} \qquad \text{and} \qquad u_2 = \frac{1 - b + \sqrt{(1-b)^2 - 4(a-b)}}{2}. \tag{5.5}$$

It follows from Figure 5.18 that u_1 and u_2 are asymptotically stable and u_0 is unstable. For

$a = b$ the equilibriums $u_0 = u_1 = 0$ and u_2 are both asymptotically stable. For $a > b$ but close to b again, there are three equilibriums: $0 = u_0 < u_1 < u_2$ defined by (5.5), see Figure 5.20. We observe that u_0 and u_2 are asymptotically stable and u_1 is unstable. But further increasing a, there is a critical parameter value when the graphs of g_1 and g_2 are tangential at a point x. This happens when

$$1 - x = \frac{a}{b+x} \qquad \text{and} \qquad \frac{d}{dx}(1-x) = \frac{d}{dx}\frac{a}{b+x}.$$

Solving the above system, we get

$$a = \frac{(b+1)^2}{4} \qquad \text{and} \qquad x = \frac{1-b}{2}.$$

So $a = a_0 := \frac{(b+1)^2}{4}$ is again a bifurcation point. For $a = a_0$, there are two equilibrium points, $u_0 = 0$ and $u_1 = u_2 = (1-b)/2$. u_0 is asymptotically stable and u_1 is unstable. And for $a > a_0$, there is only one equilibrium, which is asymptotically stable. Here $a = b$ is a transcritical, and $a = a_0$ is a saddle-node bifurcation point. See the bifurcation diagram in Figure 5.21, where the three branches of equilibrium points u_0, u_1 and u_2 are denoted by green, red and blue colors. We note that as a approaches b from below then the bifurcation diagram in Figure 5.21 approaches to Figure 5.17, where $b = a_0 = 1$. □

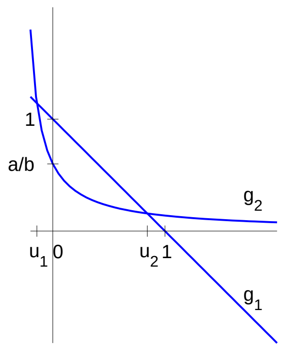

Figure 5.18: $a < b$, $b < 1$

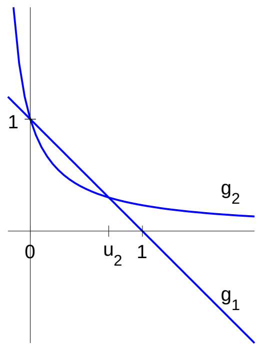

Figure 5.19: $a = b$, $b < 1$

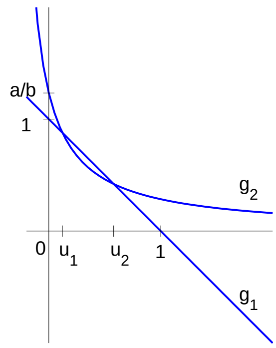

Figure 5.20: $a > b$, $b < 1$

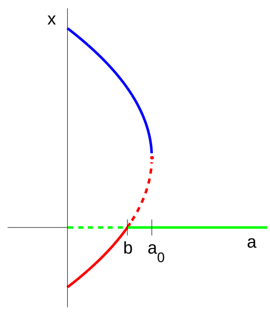

Figure 5.21: transcritical bifurcation at $a = b$, saddle-node bifurcation at $a = a_0$, $b < 1$

5.2 Two-dimensional systems

We show only two simple examples for the two-dimensional case. A more systematic study is beyond the goal of these lecture notes.

Example 5.5 Consider now a two-dimensional system

$$\begin{aligned} x' &= a - x^2, \\ y' &= -y. \end{aligned} \tag{5.1}$$

To find the equilibriums, consider the algebraic system

$$\begin{aligned} a - x^2 &= 0, \\ -y &= 0. \end{aligned}$$

Clearly, if $a < 0$, it has no solution, for $a = 0$ the only equilibrium is at $u_0 = (0,0)$, and for $a > 0$ there are two equilibriums $u_1 = (\sqrt{a}, 0)$ and $u_2 = (-\sqrt{a}, 0)$. Therefore at $a = 0$ there is again a saddle-node bifurcation. The trajectories are shown in Figures 5.22–5.24. The picture show that for $a > 0$ the equilibrium u_1 is a saddle, u_2 is an asymptotically stable node. This example was the motivation for the terminology of the saddle-node bifurcation. □

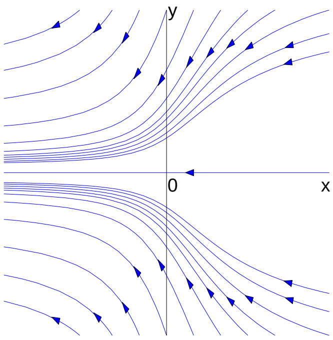

Figure 5.22: $a < 0$

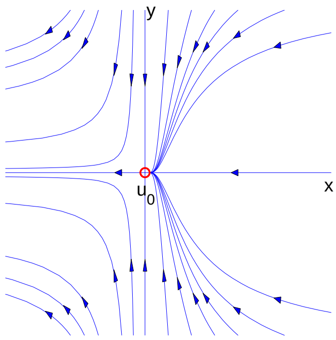

Figure 5.23: $a = 0$

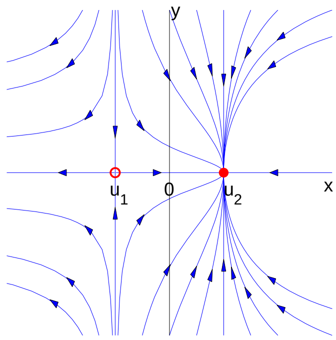

Figure 5.24: $a > 0$

Example 5.6 Consider the planar system

$$\begin{aligned} x' &= \mu x - y - x(x^2 + y^2), \\ y' &= x + \mu y - y(x^2 + y^2), \end{aligned} \tag{5.2}$$

where $\mu \in \mathbb{R}$ is a parameter.

(x, y) is an equilibrium of Eq. (5.2), if and only if

$$\mu x - y = x(x^2 + y^2), \tag{5.3}$$

$$x + \mu y = y(x^2 + y^2) \tag{5.4}$$

holds. Multiplying the first equation by x, the second by y and adding together the two equations we get

$$\mu x^2 - xy + xy + \mu y^2 = (x^2 + y^2)^2,$$

which gives

$$\mu(x^2 + y^2) = (x^2 + y^2)^2.$$

Hence (0,0) is always a solution of this equation, but for $\mu \leq 0$ there is no other solution. Therefore, for $\mu \leq 0$ there is only the trivial equilibrium of Eq. (5.2). For $\mu > 0$ we show that there is no other equilibrium, as well. It follows from the above equation that either $x^2 + y^2 = 0$, i.e., the equilibrium is $(0,0)$, or $\mu = x^2 + y^2$ with some $\mu > 0$. Substituting it into (5.3) and (5.4) we get after simplification that $y = 0$ and $x = 0$, so the contradiction yields that the only equilibrium is $(0, 0)$.

We introduce the polar coordinates r and θ defined by $x = r\cos\theta$ and $y = r\sin\theta$. Then $r^2 = x^2 + y^2$. Consider this relation along a solution $x(t)$ and $y(t)$. Then differentiating both sides with respect to t and using Eq. (5.2) we get

$$\begin{aligned} 2rr' &= 2xx' + 2yy' \\ &= 2x(\mu x - y - x(x^2+y^2)) + 2y(x + \mu y - y(x^2+y^2)) \\ &= 2\mu x^2 + 2\mu y^2 - 2(x^2+x^2)^2 \\ &= 2\mu r^2 - 2r^4. \end{aligned}$$

Therefore r satisfies the equation

$$r' = \mu r - r^3 \tag{5.5}$$

along a solution. Similarly, differentiating both sides of $\operatorname{tg}\theta = \frac{y}{x}$, we get

$$\begin{aligned} \frac{1}{\cos^2\theta}\theta' &= \frac{y'x - yx'}{x^2} \\ &= \frac{(x+\mu y - y(x^2+y^2))x - y(\mu x - y - x(x^2+y^2))}{x^2} \\ &= \frac{x^2+y^2}{x^2} \\ &= \frac{r^2}{x^2}. \end{aligned}$$

Therefore θ satisfies

$$\theta' = 1 \tag{5.6}$$

along a solution (except for the trivial solution). It follows from (5.6) that there is no other equilibrium than the trivial equilibrium for any $\mu \in \mathbb{R}$.

Its linearized equation associated to Eq. (5.2) is

$$\begin{aligned} x' &= \mu x - y \\ y' &= x + \mu y, \end{aligned} \tag{5.7}$$

whose coefficient matrix is

$$\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}.$$

The eigenvalues of it are $\lambda = \mu \pm i$. Therefore the trivial solution of Eq. (5.7) is asymptotically stable for $\mu < 0$ (see Figure 5.20 where all trajectories in a neighborhood of the origin spiral toward the origin), and it is unstable for $\mu > 0$, and hence the same property holds for the trivial solution of the nonlinear equation (5.2). Therefore $\mu = 0$ is a bifurcation point, since the stability property of the equilibrium changes. But for $\mu > 0$ some other interesting phenomenon happens: Eq. (5.5) has a positive equilibrium solution, $r = \sqrt{\mu}$. To a solution with constant r there corresponds a circle trajectory by (5.6), i.e, Eq. (5.2) has a periodic solution. It can be seen in Figure 5.26 that the trajectories approach the circle. Such a closed trajectory which attracts nearby trajectories as $t \to \infty$ or as $t \to -\infty$ is called a *limit cycle*.

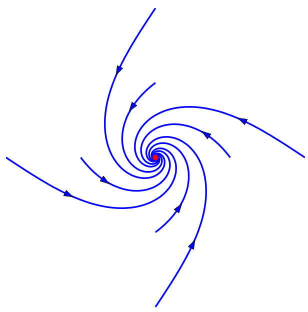

Figure 5.25: $\mu < 0$

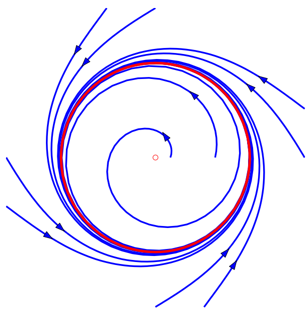

Figure 5.26: $\mu > 0$

For $\mu < 0$ Eq. (5.2) has no periodic solution, since r is strictly decreasing along a solution by (5.5). Such a bifurcation where stability of an equilibrium point is lost due to the fact that the eigenvalues of the linearized equation cross the imaginary axis is called *Hopf bifurcation*. Crossing a Hopf bifurcation point a periodic solution appears. In Eq. (5.2) the periodic solution is stable, i.e., trajectories which start close to the periodic orbit approach it as $t \to \infty$ (see Figure 5.26). □

Chapter 6
Time delays in modeling

In many dynamical models the present state of the system is determined by the past history of the system, in the simplest case, by the state of the system τ time ago, where $\tau > 0$ is a fixed constant. Such differential equations are called *delay differential equations*. In this chapter, we present two simple applications where time delay appears naturally in the model.

6.1 Self regulation population model with delayed regulation

In Example 1.21, we introduced the logistic differential equation (1.4) as a widely used model for a single species population. The logistic equation has been used successfully to model the growth of yeast cells, fruit flies, the population of Sweden and USA, the Pacific Halibut fishery and so on. But there are many experiments where we observe oscillation in the size of the population which does not appear in the classical logistic model (1.4).

We can get more complicated behavior of the solution if we introduce *time delays* in differential equations. The delays or *time lags* can represent gestation times, incubation periods, transport delays, etc., in the model, so they can be introduced naturally in biological models.

In 1948, *Hutchinson* introduced a time delay in the self-regulatory mechanisms in the logistic equation. He showed that the time lag induced oscillations of the solution which explains the same observation in some animal populations, e.g., Daphnia (water flea). Hutchinson assumed that the per capita growth rate has the form

$$\frac{N'(t)}{N(t)} = r - mN(t-\tau), \qquad t \geq 0,$$

where $r, m > 0$ and $\tau > 0$ are given constants. The constant τ is called delay or time lag. (In fact, it might be considered as the reaction time of the system.) So the simplest *delayed logistic equation* is as follows

$$N'(t) = N(t)\left(r - mN(t-\tau)\right), \qquad t \geq 0, \tag{6.1}$$

or equivalently

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right), \qquad t \geq 0; \quad K = \frac{r}{m}. \tag{6.2}$$

The derivation of (6.1) was given by Hutchinson (1948) and an other way by Cunningham (1954).

At the initial time $t = 0$, the right-hand-side of (6.2) uses the values of the solution at $-\tau$, and as $t \in [0, \tau]$, the delayed argument $t - \tau$ takes values in the interval $[-\tau, 0]$. Therefore, in order to get a unique solution of the delay equation (6.2), we have to specify the initial value of the solution on the interval $[-\tau, 0]$.

Therefore, we associate an initial condition to (6.2) in the form

$$N(t) = \varphi(t), \qquad -\tau \leq t \leq 0, \tag{6.3}$$

where $\varphi\colon [-\tau, 0] \to [0, \infty)$ is a given continuous function. φ is called *initial function*.

Solution of problem (6.2)-(6.3) can be given by using the so-called *method of steps*. Namely, if $\tau > 0$ and $t \in [0, \tau]$, then $t - \tau \in [-\tau, 0]$ and hence

$$N(t-\tau) = \varphi(t-\tau), \qquad t \in [0, \tau].$$

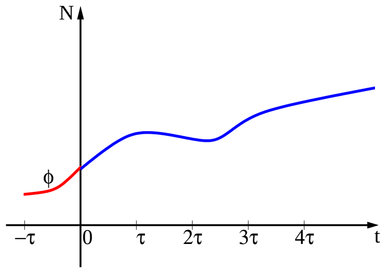

Figure 6.1: Method of steps

Thus the delayed logistic equation has the form

$$N'(t) = rN(t)\left(1 - \frac{\varphi(t-\tau)}{K}\right), \qquad t \in [0,\tau],$$

and hence

$$N(t) = N_1(t), \qquad 0 \le t \le \tau,$$

where

$$N_1(t) = \varphi(0)\exp\left(\int_0^t r\left(1 - \frac{\varphi(s-\tau)}{K}\right)\right)\mathrm{d}s, \qquad 0 \le t \le \tau.$$

In general,

$$N(t) = N_n(t), \qquad (n-1)\tau \le t \le n\tau, \qquad n \ge 1,$$

where

$$N_n(t) = N_{n-1}((n-1)\tau)\exp\left(\int_{(n-1)\tau}^t r\left(1 - \frac{N_{n-1}(s-\tau)}{K}\right)\right)\mathrm{d}s,$$

for $(n-1)\,\tau \le t \le n\tau,\;\; n \ge 1.$

The sequence of functions $N_n : [(n-1)\,\tau, n\tau] \longrightarrow \mathbb{R}$ is well-defined, and the function $N : [0,\infty) \longrightarrow \mathbb{R}$ defined by

$$N(t) = N_n(t), \qquad (n-1)\tau \le t \le n\tau, \qquad n \ge 1,$$

is the unique solution of the delayed logistic equation (6.2) with initial condition (6.3).

Now consider a special case of (6.2) with the parameter values:

$$r = 1, \qquad K = 1 \qquad \text{and} \qquad \varphi(t) = 0.5, \qquad -\tau \le t \le 0,$$

where $\tau \ge 0$ is an arbitrarily fixed parameter. In Figures 6.2–6.6, we plotted the numerical solution of the corresponding IVPs with the special parameter values

$$\tau = \frac{1}{2e} \approx 0.1839, \quad \tau = \frac{1}{e} \approx 0.3679, \quad \tau = \frac{2}{e} \approx 0.7358, \quad \tau = \frac{\pi}{2} \approx 1.571, \quad \tau = \pi \approx 3.142,$$

respectively. We observe that for small delays most of the solutions behave similarly to the solutions of the classical logistic equation, i.e., they converge monotone increasingly to the carrying capacity K. But as the time delay reaches and passes a critical value (a bifurcation point), in this case $\tau = \frac{1}{e}$, the monotonicity is lost and all solutions oscillate, but their limit is still K. Increasing the delay further, we observe a periodic solution (another bifurcation point), and for large delay the convergence is lost, the solution oscillates unboundedly.

Since the solution $N(t)$ the IVP (6.2)-(6.3) is positive whenever $\varphi(t) > 0,\;\; -\tau \le t \le 0$, the function

$$x(t) = \ln\frac{N(t)}{K} \qquad t \ge -\tau,$$

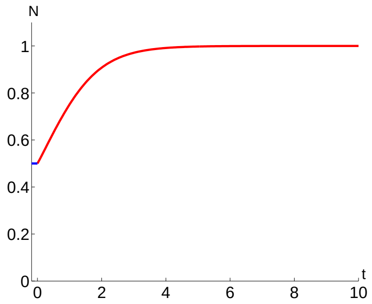

Figure 6.2: $\tau = 1/2e$

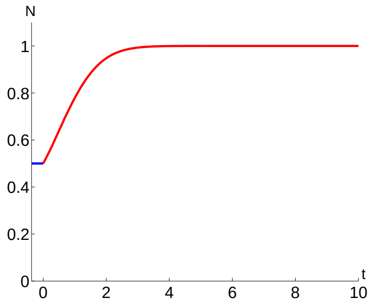

Figure 6.3: $\tau = 1/e$

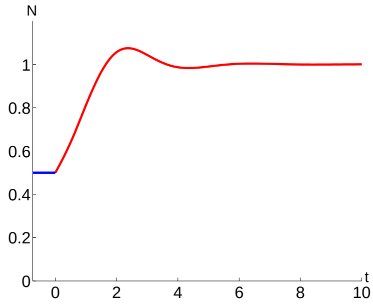

Figure 6.4: $\tau = 2/e$

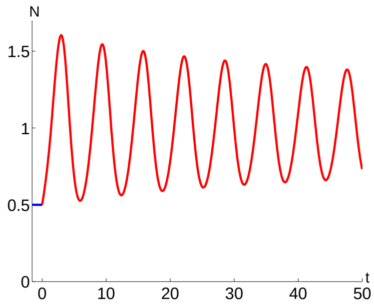

Figure 6.5: $\tau = \pi/2$

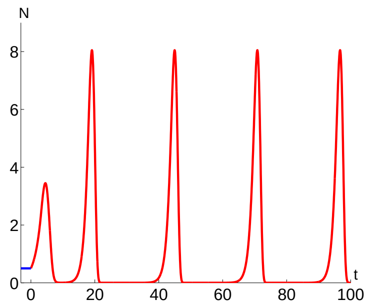

Figure 6.6: $\tau = \pi$

is well defined. Clearly, $N(t) = K\mathrm{e}^{x(t)},\ \ t \geq -\tau$.

Then

$$x'(t) = \frac{N'(t)}{N(t)} = r\left(1 - \frac{N(t-\tau)}{K}\right) = r(1 - \mathrm{e}^{x(t-\tau)}), \qquad t \geq -\tau.$$

So IVP (6.2)-(6.3) is equivalent to the IVP

$$x'(t) = r(1 - \mathrm{e}^{x(t-\tau)}), \qquad t \geq 0, \tag{6.4}$$

and

$$x(t) = \ln\frac{\varphi(t)}{K}, \qquad -\tau \leq t \leq 0. \tag{6.5}$$

Eq. (6.4) is a delay differential equation, where the initial function $\ln\frac{\varphi(t)}{K}$ is continuous on $[-\tau, 0]$, but it is not necessarily positive (in general it is not positive).

We say that a function $x: [0, \infty) \to \mathbb{R}$ is *oscillatory*, if there exist two sequences $(t_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ such that $t_n, s_n \to \infty$ as $n \to +\infty$, and $x(t_n) < 0 < x(s_n),\ \ n \geq 1$. We say that a delay differential system is oscillatory, if all non-trivial solutions are oscillatory. Otherwise, i.e., when there is at least one non-trivial solution which is not oscillatory, we say that the equation is *non-oscillatory*.

Clearly

$$x(t) > 0 \Longleftrightarrow N(t) > K$$

$$x(t) = 0 \Longleftrightarrow N(t) = K$$

and

$$x(t) < 0 \Longleftrightarrow N(t) < K.$$

So N is oscillatory (non-oscillatory) about K if and only if x is oscillatory (non-oscillatory) about zero. The solution N tends to K as $t \to +\infty$ if and only if x tends to 0 as $t \to +\infty$.

This means that the carrying capacity K attracts the solutions of Eq. (6.2) if the zero attracts the solutions of Eq. (6.4) at infinity.

It is clear that $N(t) = K,\ \ t \geq -\tau$, is a solution of Eq. (6.2) since

$$K' = 0 = rK(1 - K/K).$$

K is an equilibrium of Eq. (6.2). The other equilibrium of Eq. (6.2) is $N = 0$ (zero solution). Equation (6.4) has only one equilibrium solution, namely the zero solution. Since

$$f(x) = r(1 - \mathrm{e}^x) = -rx + r(1 - \mathrm{e}^x + x) = -rx + rx\frac{1 + x - \mathrm{e}^x}{x},$$

where

$$\frac{1 + x - \mathrm{e}^x}{x} \to 0, \text{ as } \ x \to 0,$$

the linearized version of Eq. (6.4) is as follows (see the related Theorem 4.12 for ODEs):

$$y'(t) = -ry(t - \tau), \qquad t \geq 0. \tag{6.6}$$

In the following table, we summarize conditions for stability of the trivial solution and the oscillatory property of the equation for different parameter values. It can be shown that the above properties depend on the value of the product of the coefficient r and the time delay τ in the equation:

Table 6.2: Properties of Eq. (6.6)

condition	*stability*	*oscillation*
$0 \leq r\tau \leq \frac{1}{e}$	asymptotically stable	non-oscillatory
$1/e < r\tau < \frac{\pi}{2}$	asymptotically stable	oscillatory
$r\tau = \frac{\pi}{2}$	stable, has periodic solutions	oscillatory
$r\tau > \frac{\pi}{2}$	unstable, has unbounded solution	oscillatory

In Figures 6.7–6.11, we plotted the numerical solution of the linear equation (6.6) corresponding to the constant $\varphi(t) = 0.5$ initial function and the special parameter values

$$r\tau = \frac{1}{2e}, \qquad r\tau = \frac{1}{e}, \qquad r\tau = \frac{2}{e}, \qquad r\tau = \frac{\pi}{2}, \qquad r\tau = \pi,$$

respectively. We observe the qualitative properties described in Table 6.2.

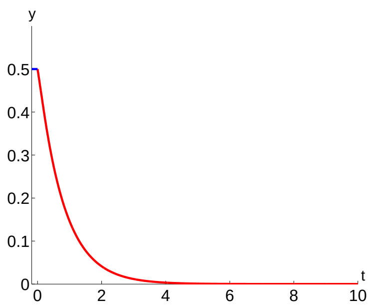

Figure 6.7: $r\tau = 1/2e$, asymptotically stable, non-oscillatory

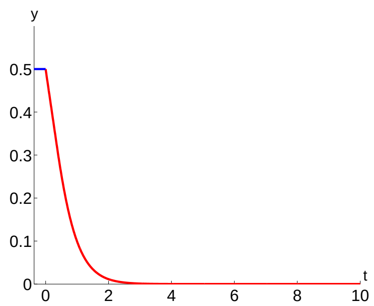

Figure 6.8: $r\tau = 1/e$, asymptotically stable, non-oscillatory

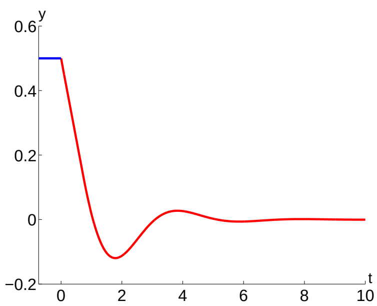

Figure 6.9: $r\tau = 2/e$, asymptotically stable, oscillatory

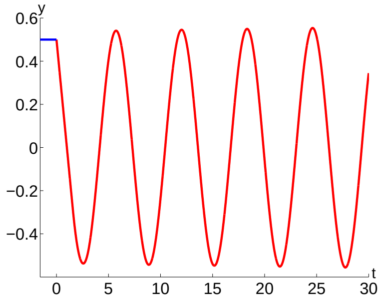

Figure 6.10: $r\tau = \pi/2$, stable, oscillatory, there is periodic solution

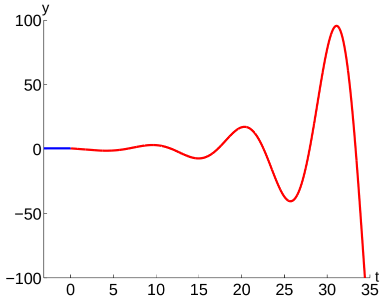

Figure 6.11: $r\tau = \pi$, unstable, oscillatory, most of the solutions are unbounded

6.2 Two connected mixing tanks model with time delay

In Example 3.17 we considered two tanks connected by two pipes (see Figure 3.1). In the derivation of the model equation (3.1) one assumption was that there is no time needed to flow the fluid through pipes (pipes were assumed to be short). Now we omit this assumption, so we assume time τ is needed for the fluid to flow through both pipes (length and width of pipes are equal). The rest of the assumptions are the same. Then at time t the outflow rate from the first tank is again $rQ_1(t)/V_1$ kg/min, but the inflow rate is $rQ_2(t-\tau)/V_2$ kg/min. Similar formulas are true for the second tank. Hence the modified model is the linear delay differential system

$$\begin{aligned} Q_1'(t) &= -r\frac{Q_1(t)}{V_1} + r\frac{Q_2(t-\tau)}{V_2}, \qquad & Q_1(0) = A_1 \\ Q_2'(t) &= r\frac{Q_1(t-\tau)}{V_1} - r\frac{Q_2(t)}{V_2}, \qquad & Q_2(0) = A_2. \end{aligned} \tag{6.1}$$

In Figure 6.12, we plotted the numerical solution of (6.1) corresponding to parameter values used in Example 3.17 and with the time delay $\tau = 15$. We observe that both solutions converge to a constant value (same as in Example 3.17), but the monotonicity of the convergence is lost due to presence of time delay.

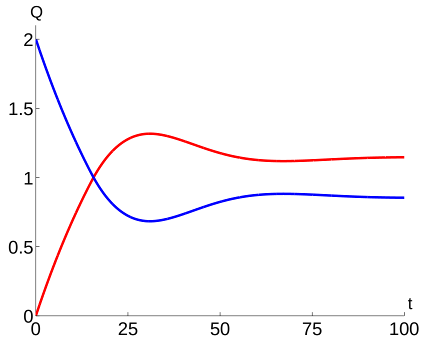

Figure 6.12: Tank model with time delay, red: $Q_1(t)$, blue: $Q_2(t)$

Generalizations of the model (6.1) can be used in many applications. For example, compartment systems or neural networks consist of units (like tanks) and data or material flow between the different units. In many cases time delay is assumed to exist between connections. For more examples of applications modeled by time delay systems, we refer to [4], [5] and [6].

Chapter 7
First-order difference equations and discrete population models

In this chapter, we show some basic solution techniques of first-order linear difference equations through discrete population models.

7.1 Introduction

Difference equations usually describe the evolution of certain phenomenon over the course of time. As an example, we shall consider *populations* with a fixed interval between generations or possibly a fixed interval between measurements. Thus we shall describe population size by a sequence $\{x(n)\}$, with $x(0) = x_0$ denoting the initial population size, $x(1)$ is the population size at the next generation (at time t_1), $x(2)$ is the population size at the next generation (at time t_2), and so on. Usually the time interval between the generations is taken to be a constant.

Assume that in certain population size of the $(n+1)$st generation $x(n+1)$ is a function of n and the size of the nth generation $x(n)$. Then the relation between $x(n+1)$ and $x(n)$ is expressed in the first-order difference equation

$$x(n+1) = f(n, x(n)), \qquad n \geq n_0, \tag{7.1}$$

under the initial condition

$$x(n_0) = x_0, \tag{7.2}$$

where $f\colon \mathbb{Z}_+ \times \mathbb{R} \to \mathbb{R}$. Here $\mathbb{Z}_+$ is the set of non-negative integers.

Eq. (7.1) is called *first-order non-autonomous* or *time-variant difference equation*. Here n_0 is the initial time, and x_0 is the initial size of the population at $n = n_0$. It can be shown easily by iteration that the IVP (7.1)-(7.2) has a unique solution $x(n) = x(n, n_0, x_0)$, $n \geq n_0$.

It is clear that

$$x(n_0 + 1) = f(n_0, x(n_0)) = f(n_0, x_0).$$

We have

$$x(n_0 + 2) = f(n_0 + 1, x(n_0 + 1)) = f(n_0 + 1, f(n_0, x(n_0))) = f(n_0 + 1, f(n_0, x_0)).$$

In a similar way, we get

$$x(n_0 + 3) = f(n_0 + 2, x(n_0 + 2)) = f(n_0 + 2, f(n_0 + 1, f(n_0, x_0))),$$

and so on.

If the function f in Eq. (7.1) does not depend on time n, i.e., if it is replaced by a function $g\colon \mathbb{R} \to \mathbb{R}$, then we have

$$x(n+1) = g(x(n)), \qquad n \geq n_0, \tag{7.3}$$

with an initial condition

$$x(n_0) = x_0. \tag{7.4}$$

Eq. (7.3) is called *time-invariant* or *autonomous* difference equation. Starting from an initial condition x_0 at time n_0, the solution of IVP (7.3)-(7.4) is given by the sequence

$$x_0,\ g(x_0),\ g(g(x_0)),\ g(g(g(x_0))), \ldots$$

We introduce the notations

$$g^0(x_0) = x_0,\ g^1(x_0) = g(x_0),\ g^2(x_0) = g(g(x_0)),\ \dots, g^n(x_0) = g(g^{n-1}(x_0)), \dots,$$

where g^n is called the *nth iterate of g*.

Now we show the construction of the model equations for *unstructured populations* in discrete time. Unstructured means that we ignore differences between individuals. We start with it because it is the easiest case to construct the model. The starting point for modeling population change is the fundamental Birth, Death, Immigration, Emigration ("BDIE") Balance Law for the total population size $x(n)$:

$$x(n+1) = x(n) + \text{Birth} - \text{Death} + \text{Immigration} - \text{Emigration}, \tag{7.5}$$

where (7.5) is always true, but it is vacuous until we specify values B, D, I and E over the time interval between t_n and t_{n+1}, Clearly, $t_n < t_{n+1}$ and $x(n)$ and $x(n+1)$ denote size of the population at time t_n and t_{n+1}, respectively.

7.2 Linear population models

To construct the first simple mode, we start from the next simplest possible assumptions:

- The population is closed, i.e., the immigration and emigration are not present ($I = E = 0$).
- The Birth and Death equal to the number of nth generation times a constant, i.e.,

$$B(n) = bx(n) \qquad \text{and} \qquad D(n) = dx(n).$$

Here constants

$$b = \frac{B(n)}{x(n)} \qquad \text{and} \qquad d = \frac{D(n)}{x(n)}, \qquad n \geq n_0$$

are called *per capita birth* and *death rate*, respectively.

Under the above conditions, the model equation is

$$x(n+1) = x(n) + (b-d)x(n) = rx(n),$$

where $r = 1 + b - d$ is the *per capita growth rate*. Clearly, $d \leq 1$, and hence $r \geq 0$. The obtained linear first-order difference equation is

$$x(n+1) = rx(n), \qquad n \geq n_0 \tag{7.1}$$

with the initial condition

$$x(n_0) = x_0, \tag{7.2}$$

where $x_0 > 0$. Simple calculation shows that the sequence

$$x(n) = x_0 r^{n-n_0}, \qquad n \geq 0$$

is the unique solution of IVP (7.1)-(7.2). Since $r > 0$, we have three possible cases:

(i) $x(n) = x_0 r^{n-n_0} \to 0$ as $n \to \infty$, if $0 < r < 1$, (or equivalently $0 \leq b < d \leq 1$);

(ii) $x(n) = x_0$, $n \geq 0$, if $r = 1$, (or equivalently $0 \leq b = d \leq 1$);

(iii) $x(n) = x_0 r^{n-n_0} \to \infty$ as $n \to \infty$, if $r > 1$, (or equivalently $b > d$).

In a more realistic case, birth rate and death rate may vary in time, and hence the per capita growth rate is also time-dependent:

$$r = r(n) \geq 0, \qquad n \geq n_0.$$

In that case, our model equation is

$$x(n+1) = r(n)x(n), \qquad n \geq n_0, \tag{7.3}$$

with the initial condition

$$x(n_0) = x_0. \tag{7.4}$$

Equation (7.3) is a *first-order linear homogeneous difference equation.* By mathematical induction, we get the following result.

Proposition 7.1 *The homogeneous equation (7.3) with initial condition (7.4) has exactly one solution given in the form*

$$x(n) = r(n-1)\cdots r(n_0)x_0 = \prod_{i=n_0}^{n-1} r(i)\cdot x_0, \qquad n > n_0. \tag{7.5}$$

Eq. (7.3) holds if either the population is closed, i.e., $I(n) = E(n) = 0$, or the *migration* $M(n) = I(n) - E(n) = 0$, $n \geq 0$. If the migration is not identically zero, then from the "BDIE" Balance Law (7.1) we get the *linear inhomogeneous* model equation

$$x(n+1) = r(n)x(n) + M(n), \qquad n \geq n_0 \tag{7.6}$$

with an initial condition

$$x(n_0) = x_0. \tag{7.7}$$

One can easily show the next result.

Proposition 7.2 *The unique solution $x(n)$ of IVP (7.3)-(7.4) is given by*

$$x(n) = \prod_{i=n_0}^{n-1} r(i)\cdot x_0 + \sum_{j=n_0}^{n-1}\left(\prod_{i=j+1}^{n-1} r(i)\right)M(j), \qquad n > n_0. \tag{7.8}$$

As a simple corollary of the above statement, we get

Proposition 7.3 *(i) If $r(n) = r$ is constant, then the solution of equation*

$$x(n+1) = rx(n) + M(n), \qquad n \geq n_0$$

with condition (7.7) is given by

$$x(n) = r^{n-n_0}x_0 + \sum_{j=n_0}^{n-1} r^{n-j-1}M(j), \qquad n > n_0. \tag{7.9}$$

(ii) If $r(n) = r$ and $M(n) = M$ both are constants, then the solution of the equation

$$x(n+1) = rx(n) + M$$

with condition (7.7) is given by

$$x(n) = \begin{cases} r^{n-n_0}x_0 + \frac{r^{n-n_0}-1}{r-1}M = \left(x_0 - \frac{M}{1-r}\right)r^{n-n_0} + \frac{1}{1-r}M, & \text{if } r \neq 1, \\ x_0 + (n-n_0)M, & \text{if } r = 1 \end{cases} \tag{7.10}$$

for $n \geq n_0$.

It is clear that negative values of $x(n)$ for population equations have no biological meaning because the size of a population is nonnegative. For this reason, our model equations are valid until the solution is nonnegative. We say that the population becomes *extinct* once there is an n_1 such that $x(n_1) \le 0$. For example the population could become extinct if the growth rate r is less than 1 and the per generation migration rate $M = I - E$ becomes negative.

Solving the next examples, we shall utilize our knowledge on homogeneous and inhomogeneous first-order difference equations.

Example 7.4 Suppose a certain population is growing at the rate of 2% per year, and the migration rate M is constant. Let $x_0 > 0$ be the size of the initial population. Formulation of the model here is based on the fact that the growth rate is $r = 1 + 0.02 = 1.02$ per year. Then the size of the population $x(n)$ in the nth year satisfy

$$x(n+1) = 1.02x(n) + M, \qquad n \ge 0. \tag{7.11}$$

Applying formula (7.10), we get the explicit solution of Eq. (7.11) as follows

$$x(n) = \left(x_0 + \frac{M}{0.02}\right) 1.02^n - \frac{M}{0.02} = 1.02^n x_0 + \frac{1.02^n - 1}{0.02} M, \qquad n \ge 0. \tag{7.12}$$

If the immigration is stronger than the emigration, i.e., the migration rate is nonnegative, then $x(n) \ge x_0 > 0$, $n \ge 0$, and the population growth unboundedly. Thus strong immigration may produce survival of the population.

Now, assume that the emigration is stronger than the immigration, i.e., $M = I - E < 0$, or equivalently $I < E$. Let $n_1 > 0$ be such that $x(n_1) < 0$, i.e.,

$$x(n_1) = 1.02^{n_1} x_0 + \frac{1.02^{n_1} - 1}{0.02} M < 0.$$

The last inequality holds if and only if

$$M < \frac{1.02^{n_1}}{1 - 1.02^{n_0}} 0.02 x_0. \tag{7.13}$$

Since the sequence $\frac{1.02^n}{1-1.02^n}$ $(n \ge 0)$ is monotone decreasing and its limit is -1, (7.13) is satisfied with an n_1 if and only if

$$I - E = M < -0.02x_0 = -(r-1)x_0.$$

This yields that strong emigration may produce the extinction of the population. □

Example 7.5 Suppose a certain population is decreasing at the rate 1% per year. Then $r = 1 - 0.01 = 0.99$, and the model equation is

$$x(n+1) = 0.09x(n) + M, \qquad n \ge 0,$$

assuming that the migration rate per year M is constant. The solution is

$$x(n) = \left(x_0 - \frac{M}{0.01}\right) 0.99^n + \frac{M}{0.01} = 0.99^n x_0 + \frac{1 - 0.99^n}{0.01} M, \qquad n \ge 0.$$

This explicit formula allows the reader to show:

(i) If $x_0 > 0$ and $M = I - E > 0$, then $x(n) > 0$, moreover the size $x(n)$ of the population approaches to $100M$ at infinity, and hence the population survives.

(ii) If $x_0 > 0$ and $M = 0$, then $x(n) > 0$ and $x(n) \to 0$ as $n \to \infty$. Hence the population becomes extinct at infinity.

(iii) If $x_0 > 0$ and $M = I - E < 0$, then the population becomes extinct at a finite time.

□

Chapter 8
Higher-order difference equations

This chapter contains some basic knowledge and some motivating applications related to higher-order linear difference equations.

8.1 Introduction

Consider the higher-order linear difference equations with constant coefficients

$$x(n) = a_1 x(n-n_1) + \cdots + a_k x(n-n_k), \qquad n \geq n_k + 1, \tag{8.1}$$

where $a_1, \ldots, a_k$ are real constants, $a_k \neq 0$, $k \geq 1$, and the positive integers $n_1, \ldots, n_k$ are such that

$$1 \leq n_1 \leq n_2 \leq \cdots \leq n_k. \tag{8.2}$$

A sequence $(x(n))_{n\geq 1}$ of real numbers is called the solution of Eq. (8.1) if it satisfies (8.1) for any $n \geq n_k + 1$. From (8.1) it is clear that the solution of (8.1) exists and it is unique if the initial values

$$x(1) = z_1, \quad \ldots, \quad x(n_k) = z_{n_k} \tag{8.3}$$

are given.

Looking for the representation of the solution of Eq. (8.1), we start with a simple procedure. We suppose that a solution of Eq. (8.1) is in the form

$$x(n) = \lambda^n, \qquad n \geq 1.$$

Then substituting this into (8.1) we get that

$$\lambda^n = a_1 \lambda^{n-n_1} + \cdots + a_k \lambda^{n-n_k}, \quad n \geq 1.$$

Assuming that $\lambda \neq 0$, we can multiply both sides of the above relation with λ^{-n+n_k}, we get

$$\lambda^{n_k} = a_1 \lambda^{n_k - n_1} + \cdots + a_{k-1} \lambda^{n_k - n_{k-1}} + a_k,$$

or equivalently

$$\lambda^{n_k} - a_1 \lambda^{n_k - n_1} - \cdots - a_{k-1} \lambda^{n_k - n_{k-1}} - a_k = 0 \tag{8.4}$$

should hold. Since $a_k \neq 0$, $\lambda = 0$ is not a solution of (8.4). Eq. (8.4) is called the *characteristic equation* of Eq. (8.1), and the related polynomial

$$p(\lambda) = \lambda^{n_k} - a_1 \lambda^{n_k - n_1} - \cdots - a_{k-1} \lambda^{n_k - n_{k-1}} - a_k \tag{8.5}$$

is called the *characteristic polynomial* of Eq. (8.1). A root of the characteristic equation is called *eigenvalue*.

From the theory of difference equations it is known that determining the roots of the characteristic equation one may give a closed formula for the solution of Eq. (8.1). The theory is similar to that of the linear differential equations, but in the next section here we show only the general solution of the second-order homogeneous linear difference equations.

8.2 Second-order linear homogeneous difference equations

Consider the second-order linear difference equation

$$x(n) = a_1 x(n-1) + a_2 x(n-2), \qquad n \geq n_0, \tag{8.1}$$

where $a_1, a_2 \in \mathbb{R}$ and $a_2 \neq 0$. This is a special case of Eq. (8.1) with $n_1 = 1$ and $n_2 = 2$. Thus in virtue of (8.4) the characteristic equation and polynomial of (8.1) are

$$\lambda^2 - a_1\lambda - a_2 = 0 \tag{8.2}$$

and

$$p(\lambda) = \lambda^2 - a_1\lambda - a_2, \tag{8.3}$$

respectively.

In terms of the *eigenvalues*, i.e., the roots of (8.2), we have three situations to contemplate.

Case A: The discriminant $d = a_1^2 + 4a_2$ of Eq. (8.2) is positive. In this case, the problem has two different eigenvalues λ_1 and λ_2 which are the roots of (8.2), namely

$$\lambda_1 = \frac{a_1 + \sqrt{a_1^2 + 4a_2}}{2} \qquad \text{and} \qquad \lambda_1 = \frac{a_1 - \sqrt{a_1^2 + 4a_2}}{2}. \tag{8.4}$$

Because $\lambda_1 \neq \lambda_2$, it can be checked easily that the general solution of Eq. (8.1) is given by

$$x(n) = c_1\lambda_1^n + c_2\lambda_2^n, \tag{8.5}$$

where c_1 and c_2 are arbitrarily fixed real numbers.

In applications we are looking for such solution of Eq. (8.1) where the initial values $x(n_0 - 1)$ and $x(n_0 - 2)$ of the solution satisfy the initial conditions

$$x(n_0 - 1) = \phi_1, \qquad x(n_0 - 2) = \phi_2. \tag{8.6}$$

Formula (8.5) is a solution of the IVP (8.1)-(8.6) if c_1 and c_2 satisfy the system

$$\left.\begin{array}{rcl} c_1\lambda_1^{n_0-1} + c_2\lambda_2^{n_0-1} & = & \phi_1 \\ c_1\lambda_1^{n_0-2} + c_2\lambda_2^{n_0-2} & = & \phi_2 \end{array}\right\}. \tag{8.7}$$

This is a linear system for c_1 and c_2, which has a unique solution since

$$\det\begin{pmatrix} \lambda_1^{n_0-1} & \lambda_2^{n_0-1} \\ \lambda_1^{n_0-2} & \lambda_2^{n_0-2} \end{pmatrix} = (\lambda_1\lambda_2)^{n_0-2}(\lambda_1 - \lambda_2) \neq 0.$$

Summary: Assume that the discriminant $d = a_1^2 + 4a_2 > 0$. Then the IVP (8.1) and (8.6) has a unique solution in the form (8.5), where c_1 and c_2 are solutions of the system (8.7).

Case B: The discriminant $d = a_1^2 + 4a_2$ of Eq. (8.2) is zero. Since $a_2 \neq 0$ we have $a_1 \neq 0$. Thus the characteristic equation has only one root

$$\lambda_1 = \frac{a_1}{a_2} \neq 0. \tag{8.8}$$

Since λ_1 is the only root of the characteristic polynomial and also

$$p'(\lambda_1) = 2\lambda_1 - a_1 = 0, \tag{8.9}$$

λ_1 is a double root of (8.2). Clearly, λ_1^n is a solution of Eq. (8.1), and beacuse of the relation (8.9) one can easily show that

$$n\lambda_1^n$$

is also a solution of Eq. (8.1). The general solution of (8.1) can be given in the form

$$x(n) = c_1\lambda_1^n + c_2 n\lambda_1^n, \tag{8.10}$$

where c_1 and c_2 are arbitrary constants. The formula (8.10) is the solution of the IVP (8.1) and (8.6) if c_1 and c_2 satisfy

$$\begin{array}{rcl} c_1\lambda_1^{n_0-1} + c_2(n_0-1)\lambda_1^{n_0-1} &=& \phi_1 \\ c_1\lambda_1^{n_0-2} + c_2(n_0-2)\lambda_1^{n_0-2} &=& \phi_2. \end{array}$$

Simple calculation shows that the above system is equivalent to

$$\left.\begin{array}{rcl} c_1 + c_2(n_0-1) &=& \lambda_1^{-n_0+1}\phi_1 \\ c_1 + c_2(n_0-2) &=& \lambda_1^{-n_0+2}\phi_2 \end{array}\right\}. \tag{8.11}$$

Obviously system (8.11) always has a unique solution.

Summary: Assume that the discriminant $d = a_1^2 + 4a_2 = 0$. Then the IVP (8.1) and (8.6) has a unique solution in the form (8.10), where c_1 and c_2 are solutions of the system (8.11).

Case C: The discriminant $d = a_1^2 + 4a_2$ of Eq. (8.2) is negative. Then the characteristic equation (8.2) has two distinct complex roots

$$\lambda_1 = \frac{a_1}{2} + \frac{\sqrt{-4a_2 - a_1^2}}{2}i \qquad \text{and} \qquad \lambda_2 = \frac{a_1}{2} - \frac{\sqrt{-4a_2 - a_1^2}}{2}i. \tag{8.12}$$

Since $a_1^2 + 4a_2 < 0$ the imaginary part $\frac{\sqrt{-4a_2-a_1^2}}{2}$ is nonzero, and $a_2 < 0$.

Rewriting λ_1 and λ_2 into exponential and trigonometric forms we get

$$\lambda_1 = r\mathrm{e}^{i\omega} = r(\cos\omega + i\sin\omega) \qquad \text{and} \qquad \lambda_2 = r\mathrm{e}^{-i\omega} = r(\cos\omega - i\sin\omega),$$

where

$$r = \left(\left(\frac{a_1}{2}\right)^2 + \left(\frac{\sqrt{-4a_2 - a_1^2}}{2}\right)^2\right)^{1/2} = \sqrt{|a_2|}, \tag{8.13}$$

Comparing the trigonometric and canonical forms, we get

$$r\cos\omega = \frac{a_1}{2} \qquad \text{and} \qquad r\sin\omega = \frac{\sqrt{-4a_2 - a_1^2}}{2} \neq 0. \tag{8.14}$$

There are two cases:

$\mathbf{C}_1$: $a_1 = 0$ and $a_2 \neq 0$. Then $r = \sqrt{|a_2|}$ and (8.14) yield

$$\cos\omega = 0 \qquad \text{and} \qquad \sin\omega > 0,$$

and hence

$$\omega = \frac{\pi}{2}.$$

$\mathbf{C}_2$: $a_1 \neq 0$ and $a_2 \neq 0$. Then $r = \sqrt{|a_2|}$ and

$$r\cos\omega = \frac{a_1}{2} \neq 0 \qquad \text{and} \qquad r\sin\omega = \frac{\sqrt{-4a_2 - a_1^2}}{2} \neq 0.$$

Thus

$$\operatorname{tg}\omega = \frac{r\sin\omega}{r\cos\omega} = \frac{\sqrt{-4a_2 - a_1^2}}{a_1},$$

and hence

$$\omega = \text{arctg}\,\frac{\sqrt{-4a_2 - a_1^2}}{a_1}.$$

As a consequence of C_1 and C_2 we get

$$\omega = \begin{cases} \text{arctg}\,\frac{\sqrt{-4a_2-a_1^2}}{a_1}, & \text{if } a_1 \neq 0, \\ \frac{\pi}{2}, & \text{if } a_1 = 0, \end{cases} \tag{8.15}$$

where $4a_2 + a_1^2 < 0$.

With r and ω defined by (8.13) and (8.15), respectively, we get

$$\lambda_1^n = r^n \mathrm{e}^{in\omega} = r^n(\cos n\omega + i \sin n\omega) \qquad \text{and} \qquad \lambda_2^n = r^n \mathrm{e}^{-in\omega} = r^n(\cos n\omega - i \sin n\omega)$$

for any integer n. Since λ_1^n and λ_2^n are solutions of Eq. (8.1), their sum and difference are also solutions of Eq. (8.1), we get that the sequences

$$\frac{\lambda_1^n + \lambda_2^n}{2} = r^n \cos n\omega \qquad \text{and} \qquad \frac{\lambda_1^n - \lambda_2^n}{2i} = r^n \sin n\omega$$

are also solutions. Thus the general solution of Eq. (8.1) can be written in the form

$$x(n) = c_1 r^n \cos n\omega + c_2 r^n \sin n\omega \tag{8.16}$$

with two arbitrary constants c_1 and c_2, where

$$r = \sqrt{|a_2|} \qquad \text{and} \qquad \omega = \begin{cases} \text{arctg}\,\frac{\sqrt{-4a_2-a_1^2}}{a_1}, & \text{if } a_1 \neq 0, \\ \frac{\pi}{2}, & \text{if } a_1 = 0. \end{cases} \tag{8.17}$$

The general solution satisfies the initial condition (8.6) if c_1 and c_2 are solutions of

$$\left.\begin{array}{rcl} c_1 r^{n_0-1}\cos(n_0-1)\omega + c_2 r^{n_0-1}\sin(n_0-1)\omega &=& \phi_1 \\ c_1 r^{n_0-2}\cos(n_0-2)\omega + c_2 r^{n_0-2}\sin(n_0-2)\omega &=& \phi_2 \end{array}\right\}.$$

This is equivalent to

$$\left.\begin{array}{rcl} c_1 \cos(n_0-1)\omega + c_2 \sin(n_0-1)\omega &=& r^{-n_0+1}\phi_1 \\ c_1 \cos(n_0-2)\omega + c_2 \sin(n_0-2)\omega &=& r^{-n_0+2}\phi_2 \end{array}\right\}. \tag{8.18}$$

The above linear system always has a unique solution since

$$\begin{aligned} \det\begin{pmatrix} \cos(n_0-1)\omega & \sin(n_0-1)\omega \\ \cos(n_0-2)\omega & \sin(n_0-2)\omega \end{pmatrix} &= \cos(n_0-1)\omega\sin(n_0-2)\omega - \sin(n_0-1)\omega\cos(n_0-2)\omega \\ &= -\sin\omega \neq 0. \end{aligned}$$

Summary: Assume that the discriminant $d = a_1^2 + 4a_2 < 0$. Then the IVP (8.1) and (8.6) has a unique solution in the form (8.16), where r and ω are defined by (8.17), and c_1 and c_2 are solutions of the system (8.18).

Next we show numerical examples for each cases.

Example 8.1 Solve the IVP

$$x(n) = 3x(n-1) + 4x(n-2), \qquad n \geq 0, \qquad x(-1) = 2, \quad x(-2) = -2.$$

This is a second-order linear homogeneous equation with characteristic equation

$$\lambda^2 - 3\lambda - 4 = 0.$$

The eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$ are distinct reals. Therefore the general solution is

$$x(n) = c_1(-1)^n + c_2 4^n.$$

Using the initial conditions, we get

$$\begin{array}{rcl} c_1(-1)^{-1} + c_2 4^{-1} & = & 2 \\ c_1(-1)^{-2} + c_2 4^{-2} & = & -2, \end{array}$$

which gives $c_1 = -2$ and $c_2 = 0$. Therefore the solution of the IVP is the sequence

$$x(n) = -2(-1)^n, \qquad n \geq -2.$$

□

Example 8.2 Solve the IVP

$$x(n) = 6x(n-1) - 9x(n-2), \qquad n \geq 0, \qquad x(-1) = 0, \quad x(-2) = 1.$$

Its characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0.$$

The eigenvalue $\lambda_1 = 3$ is a double root. Therefore the general solution is

$$x(n) = c_1 3^n + c_2 n 3^n.$$

Using the initial conditions, we get

$$\begin{array}{rcl} c_1 3^{-1} + c_2(-1)3^{-1} & = & 0 \\ c_1 3^{-2} - c_2 2 \cdot 3^{-2} & = & 1, \end{array}$$

which gives $c_1 = -9$ and $c_2 = -9$. Therefore the solution of the IVP is the sequence

$$x(n) = -9 \cdot 3^n - 9n3^n, \qquad n \geq -2.$$

□

Example 8.3 Solve the IVP

$$x(n) = 2x(n-1) - 4x(n-2), \qquad n \geq 0, \qquad x(-1) = 2, \quad x(-2) = -1.$$

Its characteristic equation is

$$\lambda^2 - 2\lambda + 4 = 0.$$

The eigenvalues are $\lambda_1 = 1 + \sqrt{3}i$ and $\lambda_2 = 1 - \sqrt{3}i$. The trigonometric form of the eigenvalue is $\lambda_1 = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, therefore the general solution has the form

$$x(n) = c_1 2^n \cos\left(\frac{\pi}{3}n\right) + c_2 2^n \sin\left(\frac{\pi}{3}n\right).$$

Using the initial conditions we get

$$\begin{array}{rcl} c_1 2^{-1}\cos(-\pi/3) + c_2 2^{-1}\sin(-\pi/3) & = & 2 \\ c_1 2^{-2}\cos(-2\pi/3) + c_2 2^{-2}\sin(-2\pi/3) & = & -1, \end{array}$$

which gives $c_1 = 8$ and $c_2 = 0$. Therefore the solution of the IVP is the sequence

$$x(n) = 8 \cdot 2^n \cos\left(\frac{\pi}{3}n\right), \qquad n \geq -2.$$

□

8.3 Application of higher-order difference equations

Teletype and telegraphy are two examples of *discrete channel* for transmitting information. Generally, a discrete channel will mean a system where by a sequence of choices from a finite set of elementary symbols $S_1, \dots, S_k$ can be transmitted from one point to another. Each of the symbols S_i is assumed to have a certain duration in time n_i seconds (not necessary the same for different S_i, for example dots and dashes in telegraphy). Messages are transmitted by first encoding them into string or sequences of these symbols. Let $N(n)$ be the number of possible message sequences of duration n. In fact, the transmission of each of these messages requires exactly n units of time.

In the general case with different length and symbols and constraints on the allowed sequences, the following definition is introduced in [8]. The *capacity* C of a discrete channel is given by

$$C = \lim_{n\to\infty} \frac{\log_2 N(n)}{n} \tag{8.1}$$

if this limit is a finite number, where $N(n)$ is the number of allowed signals (which are represented with message sequences) of duration n, and $\log_2$ denotes the logarithm base 2. It is known that the limit in question will exist as a finite number in most of the cases of interest.

Now we discuss a case when the limit in the above definition exists as a finite number. Suppose all sequences of the symbols $S_1, S_2, \dots, S_k$ are allowed and the symbols have durations $1 \le n_1 \le n_2 \le \cdots \le n_k$. What is the channel capacity?

$N(n)$, the number of sequences of duration n is equal to the sum of the number of sequences ending with $S_1, \dots, S_k$, and these are $N(n-n_1), \dots, N(n-n_k)$, respectively, for any $n \ge n_k+1$. Thus the sequence $N(n)$ satisfies the higher-order difference equation

$$N(n) = N(n-n_1) + \cdots + N(n-n_k), \qquad n \ge n_k + 1 \tag{8.2}$$

with initial conditions

$$N(1) = u_1, \quad \dots, \; N(n_k) = u_{n_k}. \tag{8.3}$$

Looking for a solution of (8.2) in the form λ^n, we get that λ^n obeys

$$\lambda^n = \lambda^{n-n_1} + \cdots + \lambda^{n-n_k},$$

or equivalently,

$$\lambda^{n_k} - \lambda^{n_k-n_1} - \cdots - \lambda^{n_k-n_{k-1}} - 1 = 0. \tag{8.4}$$

As in the previous section, Eq. (8.4) is called the *characteristic equation* of (8.2), and its roots are called *eigenvalues*. The *characteristic polynomial* of Eq. (8.2) is

$$p(\lambda) = \lambda^{n_k} - \lambda^{n_k-n_1} - \cdots - \lambda^{n_k-n_{k-1}} - 1. \tag{8.5}$$

Let us consider an interesting special case when we have only two symbols S_1 and S_2. Say S_1 is the symbol of a dot which requires exactly 1 unit of time ($n_1 = 1$), and S_2 is the symbol of a dash which requires exactly two units of time ($n_2 = 2$).

Our goal is to find the channel capacity. In virtue of the general equation (8.2), we have that the number of sequences $N(n)$ of the same duration n satisfies

$$N(n) = N(n-1) + N(n-2), \qquad n \ge 3, \tag{8.6}$$

with an initial condition

$$N(1) = 1 \qquad \text{and} \qquad N(2) = 2. \tag{8.7}$$

In fact, only S_1 can be transmitted during one unit of time, and only S_1S_1 and S_2 are the sequences which can be transmitted during two units of time.

The related characteristic equation to Eq. (8.6) is

$$\lambda^2 - \lambda - 1 = 0,$$

which has two distinct real roots

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Thus the solution of the IVP (8.6)-(8.7) is

$$N(n) = c_1\lambda_1^n + c_2\lambda_2^n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where c_1 and c_2 satisfy the system

$$\begin{array}{rcl} c_1\lambda_1 + c_2\lambda_2 & = & 1 \\ c_1\lambda_1^2 + c_2\lambda_2^2 & = & 2. \end{array} \tag{8.8}$$

Multiplying both sides of the first equation with λ_1 and after subtracting it from the second equation, we get

$$(\lambda_2^2 - \lambda_1\lambda_2)c_2 = 2 - \lambda_1,$$

and hence

$$c_2 = \frac{2-\lambda_1}{\lambda_2(\lambda_2-\lambda_1)}.$$

By the definition of λ_1 and λ_2 we get $\lambda_2 - \lambda_1 = -\sqrt{5}$ and $2 - \lambda_1 = \frac{3-\sqrt{5}}{2}$, and hence

$$\frac{2-\lambda_1}{\lambda_2} = \frac{3-\sqrt{5}}{1-\sqrt{5}} = \frac{(3-\sqrt{5})(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} = \frac{-2+2\sqrt{5}}{-4} = \lambda_2.$$

Thus

$$c_2 = -\lambda_2\frac{1}{\sqrt{5}}.$$

Substituting this into the first equation in (8.8), we get

$$c_1 = \frac{1-c_2\lambda_2}{\lambda_1} = \frac{1+\lambda_2^2\frac{1}{\sqrt{5}}}{\lambda_1} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}+\lambda_2^2}{\lambda_1} = \frac{1}{\sqrt{5}}\lambda_1.$$

Substituting c_1 and c_2 into the formula of the solution we get that

$$N(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, \quad n \geq 1.$$

Now we are looking for the capacity of the channel in this special case. From the latest formula, we get

$$N(n) = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \frac{1}{\sqrt{5}}\left(1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1}\right),$$

where

$$\left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right| = \frac{\sqrt{5}-1}{1+\sqrt{5}} < 1.$$

Therefore

$$\frac{2\sqrt{5}}{3+\sqrt{5}} = 1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^2 < 1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1} \le 1 - \frac{1-\sqrt{5}}{1+\sqrt{5}} = \frac{2\sqrt{5}}{1+\sqrt{5}}, \qquad n \ge 1,$$

and hence

$$\begin{aligned}
\frac{\log_2 N(n)}{n} &= \frac{1}{n}\log_2\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\frac{1}{\sqrt{5}}\left(1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1}\right)\right] \\
&= \frac{1}{n}\log_2\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \frac{1}{n}\log_2\frac{1}{\sqrt{5}} + \frac{1}{n}\log_2\left(1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1}\right) \\
&= \frac{n+1}{n}\log_2\frac{1+\sqrt{5}}{2} + \frac{1}{n}\log_2\frac{1}{\sqrt{5}} + \frac{1}{n}\log_2\left(1-\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^{n+1}\right)
\end{aligned}$$

has the limit

$$C = \lim_{n\to\infty}\frac{\log_2 N(n)}{n} = \log_2\frac{1+\sqrt{5}}{2}.$$

We note that $\frac{1+\sqrt{5}}{2}$ is the greatest real eigenvalue of the equation (8.6).

Now we turn back to the calculation of the channel capacity under the next condition.

Condition A: All sequences of the symbols $S_1, \ldots, S_k$ are allowed and the symbols have duration $n_1 \le n_2 \le \cdots \le n_k$, respectively, where n_i $(1 \le i \le k)$ and $k \ge 2$ are positive integers.

Earlier in this section, we showed that under Condition A, if $N(n)$ represents the number of sequences of duration n then $N(n)$ satisfies Eq. (8.2), its characteristic equation is (8.4), and the characteristic polynomial p has the form (8.5). Clearly, 0 is not a root of p. We rewrite p in the form

$$p(\lambda) = \lambda^{n_k}(1 - \lambda^{-n_1} - \cdots - \lambda^{-n_k}).$$

Therefore, p is strictly monotone increasing on the interval $(0,\infty)$ since it is a product of two stricly monotone increasing functions on $(0,\infty)$, the functions λ^{n_k} and $1 - \lambda^{-n_1} - \cdots - \lambda^{-n_k}$. On the other hand,

$$p(1) = 1 - k\cdot 1 < 0, \qquad \text{and} \qquad \lim_{\lambda\to\infty} p(\lambda) = \infty.$$

We have therefore the following result.

Proposition 8.4 *Assume Condition A. Then the characteristic polynomial p defined by (8.5) has exactly one positive real root λ_p, and it satisfies $\lambda_p > 1$.*

Using a much more sophisticated calculation we may get the following theorem.

Theorem 8.5 *Assume Condition A, and suppose the positive real root λ_p of the characteristic polynomial p defined by (8.5) is a simple root. Then*

(i) $\lambda_p > 1$ and every (complex or real) root λ of (8.4) obeys

$$|\lambda| < \lambda_p;$$

(ii) the channel capacity C exists and it satisfies

$$C = \log_2 \lambda_p. \tag{8.9}$$

Example 8.6 Now we illustrate the above theorem. We have 6 symbols, $S_1, \ldots, S_6$ with durations $2, 4, 5, 7, 8$ and 10, respectively. We assume that all sequences of the signals are allowed. The difference equation with respect to $N(n)$ is

$$N(n) = N(n-2) + N(n-4) + N(n-5) + N(n-7) + N(n-8) + N(n-10), \qquad n \geq 11,$$

and its characteristic equation is

$$\lambda^{10} - \lambda^8 - \lambda^6 - \lambda^5 - \lambda^3 - \lambda^2 - 1 = 0.$$

Then Theorem 8.5 is applicable. Numerical approximation shows that $\lambda_p \approx 1.4529$, and hence

$$C = \log_2 \lambda_p \approx 0.53894.$$

□

In the case, when $N(n) = 0$ for arbitrary large n the definition of capacity in the form of (8.1) is not defined, and this example also demonstrates that formula (8.9) in Theorem 8.5 may hold without the simplicity assumption of the positive eigenvalue.

Example 8.7 Consider the teletype where all symbols are of the same duration say n_1 unit of time, and any sequence of the 32 symbols are allowed. In this case, the following heuristic argument gives us the channel capacity without using the difference equation.

Each symbol represent five bits of information. If the system transmits m symbols per second, i.e., the duration of each symbol is $n_1 = \frac{1}{m}$ seconds, it is natural to say that the channel has a capacity of $5m$ bits per seconds, as it is given by formula

$$C = \frac{5}{n_1}.$$

This does not mean that the teletype channel will always be transmitting information at this rate. This is the maximum possible rate and whether or not the actual rate reaches this maximum depends on the source of information which feeds the channel.

In this case $n_1 = \cdots = n_{32}$, $k = 32$, and Eq. (8.2) reduces to

$$N(n) = 32N(n - n_1), \qquad n \geq n_1 + 1.$$

Suppose for definiteness that n_1 is even. Then for all odd n there is no message of length odd, so $N(n) = 0$ for all odd n. Therefore the capacity of the channel cannot be defined by formula (8.1) in this case.

The characteristic equation is

$$\lambda^{n_1} = 32,$$

and its positive root is $\lambda_p = \sqrt[n_1]{32}$. If we compute formula (8.9) we get

$$\log_2 \lambda_p = \log_2 \sqrt[n_1]{32} = \log_2 2^{5/n_1} = \frac{5}{n_1}, \tag{8.10}$$

so it gives the capacity of the channel in this case, too, despite the fact that the multiplicity of λ_p is 32. □

Chapter 9
Stability theory for difference equations

In this chapter, we present some basic results and definitions for stability and bifurcation theory for difference equations.

9.1 Linear difference equations

Consider again the higher-order scalar linear difference equation

$$x(n) = a_1 x(n - n_1) + \cdots + a_k x(n - n_k), \qquad n \geq n_k + 1, \tag{9.1}$$

where $a_1, \ldots, a_k$ are real constants, $a_k \neq 0$, $k \geq 1$, and we associate the IC

$$x(1) = z_1, \quad \ldots, \quad x(n_k) = z_{n_k} \tag{9.2}$$

to (9.1). We have seen in the previous chapter that the characteristic equation of (9.1) has the form

$$\lambda^{n_k} - a_1 \lambda^{n_k - n_1} - \cdots - a_{k-1} \lambda^{n_k - n_{k-1}} - a_k = 0, \tag{9.3}$$

and its characteristic polynomial $p(\lambda)$ is defined by (8.5).

We say that the trivial (constant $x(n) = 0$) solution of Eq. (9.1) is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|z_1| < \delta, \quad |z_2| < \delta, \quad \ldots, \quad |z_{n_k}| < \delta$$

implies

$$|x(n)| < \varepsilon, \qquad n \geq n_k + 1.$$

We say that the trivial solution of Eq. (9.1) is *asymptotically stable* if it is stable and there exists $\sigma > 0$ such that

$$\lim_{n \to \infty} x(n) = 0$$

for all solutions corresponding to IC (9.2) with

$$|z_1| < \sigma, \quad |z_2| < \sigma, \quad \ldots, \quad |z_{n_k}| < \sigma.$$

If the trivial solution is not stable, we say that it is *unstable*.

We have the following results which are the discrete analogue of Theorems 4.9 and 4.10.

Theorem 9.1 *The trivial solution of Eq. (9.1) is*

(a) stable if and only if all solutions of Eq. (9.1) are bounded;

(b) asymptotically stable if and only if all solutions x of Eq. (9.1) satisfy

$$\lim_{n \to \infty} x(n) = 0.$$

Theorem 9.2 *Let $\lambda_1, \ldots, \lambda_{n_k}$ be the eigenvalues of Eq. (9.1), i.e., the roots of (9.3). Then the trivial solution of Eq. (9.1) is* *(a) stable if and only if*

$$|\lambda_j| \le 1, \qquad j = 1, \dots, n_k,$$

and if $|\lambda_j| = 1$ *for some* j*, then it is a simple root of (9.3), i.e.,* $p'(\lambda_j) \neq 0$*;*

(b) asymptotically stable if and only if

$$|\lambda_j| < 1, \qquad j = 1, \dots, n_k.$$

Example 9.3 Find the stability of the trivial solution of the second-order scalar equation

$$x(n) = \frac{3}{2}x(n-1) - \frac{1}{2}x(n-2), \qquad n \ge 2.$$

Its characteristic equation is

$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0,$$

which gives $\lambda = 1$ and $\lambda = \frac{1}{2}$. Therefore Theorem 9.2 yields that the trivial solution of the difference equation is stable but it is not asymptotically stable. □

9.2 First-order nonlinear scalar autonomous difference equations

Consider the first-order nonlinear scalar autonomous difference equation

$$x(n) = f(x(n-1)), \qquad n \ge 1 \tag{9.1}$$

with the associated IC

$$x(0) = x_0, \tag{9.2}$$

where $x_0 \in \mathbb{R}$.

We say that u is an *equilibrium* or *constant steady state* of Eq. (9.1) if $x(n) = u$ is a constant solution of Eq. (9.1), i.e.,

$$u = f(u). \tag{9.3}$$

Other words, we say that u is a *fixed point* of Eq. (9.1). Eq. (9.3) is called a *fixed point equation.*

We say that an equilibrium u of (9.1) is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x_0 - u| < \delta$, then corresponding solution x if the IVP (9.1)-(9.2) satisfies $|x(n) - u| < \varepsilon$ for all $n \ge 0$. The equilibrium u is *asymptotically stable* if it is stable and there exists $\sigma > 0$ that if $|x_0 - u| < \sigma$ then the corresponding solution of the IVP (9.1)-(9.2) satisfies

$$\lim_{n\to\infty} x(n) = u.$$

One way to graph solutions of a difference equation is the so-called *cobweb diagram*. Here the graph of the function f (red curve) and the graph of the identity function (magenta line) is depicted. Then the sequence $x(n)$ is depicted starting from an initial value x_0, see Figures 9.1 and 9.2. The next element of the sequence is $x(1) = f(x_0)$, and then $x(2) = f(x(1))$, and so on. Blue dots on the graph correspond to $(x_0, 0)$ and $(x(i), x(i+1))$, $i = 1, 2, \dots$. So the first and second coordinates of these points are values of the solution sequence. The fixed point u of f corresponds to the point (u, u) where the graph of f and the graph of the identity function intersect each other (magenta circle). The solution converges to the equilibrium u if blue dots on the graph of f converge to the point (u, u). We can see in Figure 9.1 that the solution tends to u monotonically if we start the sequence close enough to u and $0 < f'(u) < 1$. Figure 9.2 shows the case when $-1 < f'(u) < 0$. In this case, the solution again converges to u, but the convergence is oscillatory around u. In the latter case the blue stairs spiral around (u, u) and approach to it. In both cases, the equilibrium is asymptotically stable.

Figure 9.3 shows the case when $f'(u) > 1$. We can see that the solution gets farther from the equilibrium as time increases, so in this case the equilibrium is unstable. Figure 9.4 shows the case when $f'(u) < -1$. In this case the stairs spiral away from (u, u), so the equilibrium is unstable.

Motivated by the previous case we can formulate the following result, which can be proved rigorously too.

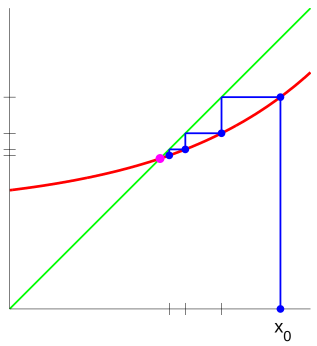

Figure 9.1: $0 < f'(u) < 1$

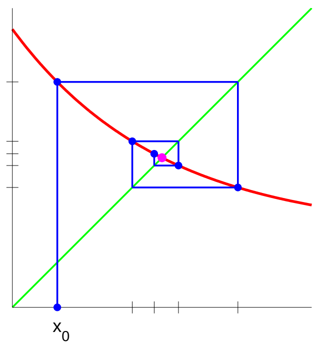

Figure 9.2: $-1 < f'(u) < 0$

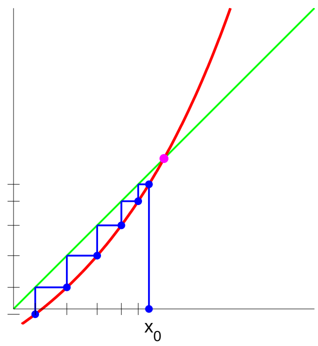

Figure 9.3: $1 < f'(u)$

Theorem 9.4 *Let u be an equilibrium of Eq. (9.1) and suppose f is continuously differentiable in a neighborhood of u. Then*

(i) if $|f'(u)| < 1$, then u is an asymptotically stable equilibrium, and

(ii) if $|f'(u)| > 1$, then u is an unstable equilibrium.

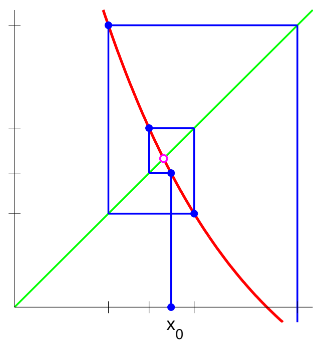

Figure 9.4: $f'(u) < -1$

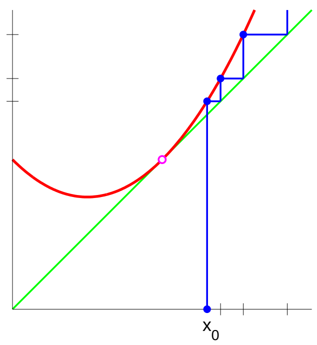

Figure 9.5: $f'(u) = 1,\ \ f''(u) > 0$

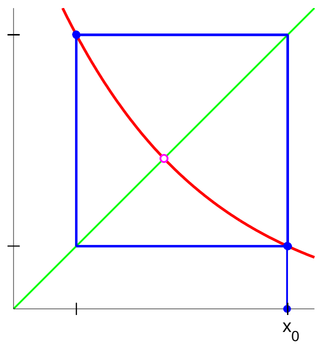

Figure 9.6: period 2 solution

In the critical case, when $|f'(u)| = 1$ higher-order derivatives can determine the stability of the equilibrium. We have the following result.

Theorem 9.5 *Let u be an equilibrium of Eq. (9.1) and suppose f is three times continuously differentiable in a neighborhood of u, and $|f'(u)| = 1$. Then*

(i) if $f'(u) = 1$ and $f''(u) \neq 0$, then u is unstable;

(ii) *if* $f'(u) = 1$, $f''(u) = 0$ *and* $f'''(u) > 0$ *then* u *is unstable;*

(iii) *if* $f'(u) = 1$, $f''(u) = 0$ *and* $f'''(u) < 0$ *then* u *is asymptotically stable;*

(iv) *if* $f'(u) = -1$ *and* $-2f'''(u) - 3(f''(u))^2 < 0$ *then* u *is asymptotically stable;*

(v) *if* $f'(u) = -1$ *and* $-2f'''(u) - 3(f''(u))^2 > 0$ *then* u *is unstable.*

Note that part (i) of the above theorem is illustrated in Figure 9.5.

Finally, we investigate periodic solutions of (9.1). $x(n)$ is a *periodic solution* of Eq. (9.1) with period p if

$$x(n+p) = x(n), \qquad n \geq 0.$$

For a 2-periodic case the solution of (9.1) has the form

$$x_0, x_1, x_0, x_1, \ldots,$$

where $x_1 = x(1)$. In Figure 9.6, a two-periodic solution is illustrated in a cobweb diagram. Then, clearly, both x_0 and x_1 are equilibriums of

$$y(n) = f(f(y(n-1))), \qquad n \geq 1. \tag{9.4}$$

The opposite case can be proved easily, thus we get the next result.

Proposition 9.6 *Let* $x(n)$ *be the solution of the IVP (9.1)-(9.2). Then* $x(n)$ *is a* p*-periodic solution of Eq. (9.1) if and only if* x_0 *is an equilibrium of the difference equation*

$$y(n) = f^p(y(n-1)), \qquad n \geq 1,$$

where f^p *is the pth iterate of the function* f.

We say that a *periodic solution* $x(n)$ of (9.1) corresponding to IC (9.2) is *asymptotically stable/unstable* if the equilibrium x_0 of Eq. (9.4) is asymptotically stable/unstable.

Example 9.7 Find all 2-periodic solutions of the difference equation

$$x(n) = -x(n-1)^2 + 1.$$

We consider the second iterate function of f:

$$f^2(x) = f(f(x)) = -(-x^2+1)^2 + 1 = -x^4 + 2x^2.$$

Its equilibriums are the solutions of

$$-u^4 + 2u^2 = u,$$

which has four solutions $u_1 = 0$, $u_2 = 1$, $u_3 = \frac{-1-\sqrt{5}}{2}$ and $u_4 = \frac{-1+\sqrt{5}}{2}$. It is easy to check that u_3 and u_4 are equilibriums of f as well, hence the solutions starting from u_3 and u_4 are constant. So these initial values do not give rise to a nontrivial 2-periodic solution. But it is easy to see that the solution starting from u_1 is 2-periodic:

$$0, 1, 0, 1, \ldots,$$

and hence the initial value $u_2 = 1$ generates essentially the same sequence.

Therefore the given difference equation has essentially one 2-periodic solution.

Now check the stability property of the 2-periodic solution. For this, compute the derivative of the 2nd iterate function $g := f^2$:

$$g'(x) = (-x^4 + 2x^2)' = -4x^3 + 4x.$$

We have $g'(0) = 0$, so by Theorem 9.4, the zero solution of the second iterate equation $y(n) = f(f(y(n-1))$ is asymptotically stable, and therefore the 2-periodic solution $0, 1, 0, 1, \ldots$ is also asymptotically stable. Similarly, $g'(1) = 0$, so the sequence $1, 0, 1, 0, \ldots$ is also asymptotically stable.

It is easy to check that equilibriums u_3 and u_4 of the difference equation are both unstable. □

9.3 Bifurcation and chaos in difference equations: discrete logistic equation

We consider the discrete analogue of the classical logistic differential equation, the so-called *discrete logistic equation*

$$x(n) = rx(n-1)\left(1 - \frac{x(n-1)}{K}\right), \qquad n \geq 1. \tag{9.1}$$

Dividing both sides of (9.1) by K and introducing $y(n) = x(n)/K$ we get

$$y(n) = ry(n-1)(1 - y(n-1)), \qquad n \geq 1. \tag{9.2}$$

This contains only a single parameter r. From the biological meaning of the model, we assume $r > 0$, and we are interested in only nonnegative solutions of this equation. Eq. (9.2) has the form of (9.1) with $f(x) = rx(1-x)$.

We investigate the stability properties of the equilibriums and periodic solutions of (9.2). It is easy to check that Eq. (9.2) has two equilibriums,

$$u_1 = 0 \qquad \text{and} \qquad u_2 = \frac{r-1}{r}.$$

First, consider the equilibrium u_1. We have $f'(x) = r - 2rx$, so $f'(u_1) = f'(0) = r$. Therefore, if $0 < r < 1$, then the equilibrium $u_1 = 0$ is asymptotically stable by Theorem 9.4. But as r passes 1, the equilibrium u_1 becomes unstable. Hence $r = 1$ is a bifurcation point.

Now consider the equilibrium u_2. If $0 < r < 1$ then $u_2 < 0$, so the population model is not defined. Hence only 1 nonnegative equilibrium exists. If r passes 1 the equilibrium becomes positive, i.e., the second equilibrium appears in the equation. The second equilibrium is asymptotically stable if

$$f'(u_2) = r - 2ru_2 = 2 - r \in (-1, 1).$$

Hence if $1 < r < 3$, u_2 is asymptotically stable, and for $r > 3$ the equilibrium u_2 becomes unstable. $r = 3$ is again a bifurcation point.

Next we examine whether Eq. (9.2) has 2-periodic solutions. For this we have to find the equilibrium of the second iterate function:

$$r^2x(1-x)(1 - rx(1-x)) = x.$$

With a tedious computation, we get the solutions

$$u_1 = 0, \quad u_2 = \frac{r-1}{r}, \quad u_3 = \frac{r+1-\sqrt{(r-3)(r+1)}}{2r} \quad \text{and} \quad u_4 = \frac{r+1+\sqrt{(r-3)(r+1)}}{2r}.$$

Since u_1 and u_2 are fixed points of f, they do not generate nontrivial 2-periodic solutions. If $r < 3$ then u_3 and u_4 are not real, but for $r > 3$ they are reals. Since u_3 and u_4 are not fixed points of f, they generate 2-periodic solutions: $u_3, u_4, u_3, u_4, \ldots$, so there is essentially one 2-periodic solution of Eq. (9.2).

To test the stability of the periodic solution, we first compute the derivative of the second iterate function $g := f^2$:

$$g'(x) = (f(f(x)))' = f'(f(x))f'(x),$$

so we have

$$g'(u_3) = f'(f(u_3))f'(u_3) = f'(u_4)f'(u_3).$$

Hence if $-1 < f'(u_4)f'(u_3) < 1$, then the 2-periodic solution is asymptotically stable. One can check that this holds if $3 < r < 1+\sqrt{6}$. If r passes $1+\sqrt{6} \approx 3.4495$, then the 2-periodic solution becomes unstable. Hence $r_1 := 1+\sqrt{6}$ is a bifurcation value.

One can show that if $r > r_1$, then Eq. (9.2) has a 4-periodic solution, which is asymptotically stable for $r_1 < r < r_2$ with $r_2 \approx 3.5441$, and it is unstable for $r > r_2$. And it can be continued to show that passing r_2 an 8-periodic solution appears, and so on. There is a sequence of parameters r_k where a period doubling bifurcation occurs, and the monotone increasing sequence r_k has a finite limit close to 3.57.

In Figures 9.7–9.9, we plotted the solutions of (9.2) for different parameter values. In Figure 9.7, the solution corresponds to $r = 2$ and $x_0 = 0.85$ can be seen. We can observe the convergence of the solution to the positive equilibrium u_2. In Figure 9.8, we generated a solution with $r = 3.3$ and $x_0 = 0.15$. We plotted the first 20 terms of the solution, but we can observe that after a few terms two values are repeated in the sequence, so the solution approaches the 2-periodic solution. In Figure 9.9, the parameter value $r = 3.9$ is used. We generated the first 150 terms of the sequence. It looks that the values in the sequence are "random" values without any regularity. Such a behavior is called *chaotic*. The definition of *chaos* is not universally accepted yet in the mathematical literature, but in almost all definitions, it is common that a chaotic behavior is aperiodic and it is sensitive for small changes in the initial conditions.

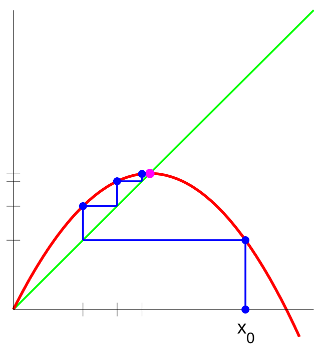

Figure 9.7: $r = 2$, $x_0 = 0.85$

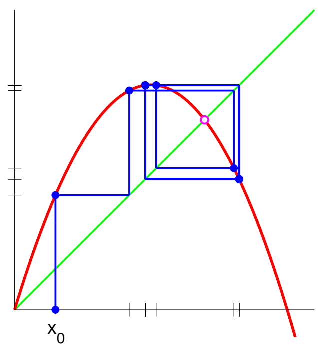

Figure 9.8: $r = 3.3$, $x_0 = 0.15$

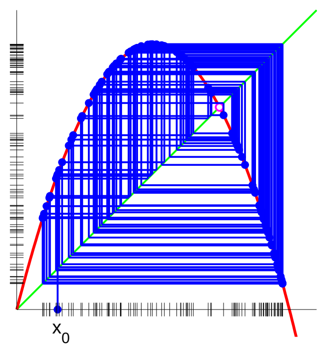

Figure 9.9: $r = 3.9$, $x_0 = 0.15$

Chapter 10
Hybrid systems and a control application

In this section, we show a control problem where a so-called *hybrid system*, i.e., a system with continuous and discrete arguments appears naturally in the model. Moreover, in this application a *time delay* is introduced into the model.

10.1 Problem formulation

We study the problem of adjusting the concentration of salt to a desired level in a mixing tank (see Figure 10.1). We assume that at time $t = 0$, the tank contains V liters of solution with an initial salt concentration of c_0 grams per liter. Salt concentration in the incoming fluid is s gram per liter. We suppose that the incoming fluid is immediately mixed thoroughly, so we assume the concentration in the tank is uniform.

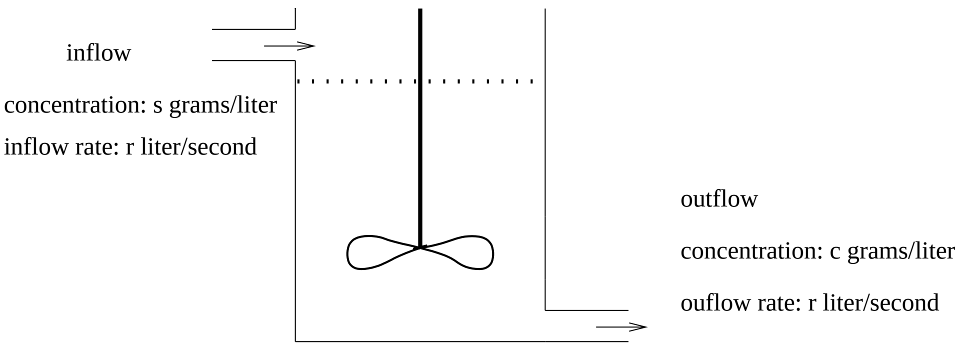

Figure 10.1: a mixing tank

Our goal is to find the incoming concentration of salt s, so that the concentration of salt in the tank attains (and remains at) a predetermined concentration k.

The volume of the solution at the tank is constant V, since inflow and outflow rates are the same. The rate of change of the mass of salt equals to the rate it enters minus the rate at which salt leaves the tank. The rate of change salt enters to the tank is s g/l $\cdot$ r l/sec=sr g/sec, and the rate of change salt leaves the tank is $c(t)$ g/l $\cdot$ r l/sec=rc(t) g/sec. The mass of salt at time t is $Vc(t)$. Therefore

$$\frac{d}{dt}(Vc(t)) = \text{inflow rate} - \text{outflow rate} = sr - c(t)r,$$

so the concentration c satisfies the first-order inhomogeneous linear differential equation

$$c'(t) + pc(t) = ps, \qquad t \geq 0 \tag{10.1}$$

where

$$p = \frac{r}{V}$$

is a constant, and the initial condition is $c(0) = c_0$. The solution of the IVP is

$$c(t) = s + (c_0 - s)\,\mathrm{e}^{-pt} \qquad t \geq 0.$$

From the above formula, it is clear that $c(t) \to s$ monotone decreasingly for $c_0 > s$ and monotone increasingly for $c_0 < s$. Therefore if $s \neq k$ then $c(t)$ either never gets close to k or $c(t)$ reaches

k in finite time, but then the solution gets far from the level k. If $s = k$ then the solution approaches k, but never reaches it in finite time. So our goal cannot be accomplished under the above circumstances.

Next, we suppose that we have a device which can control the concentration s of the inflow fluid depending on the instantaneous concentration of the solution in the tank (see Figure 10.2). Such a control mechanism is called *feedback*.

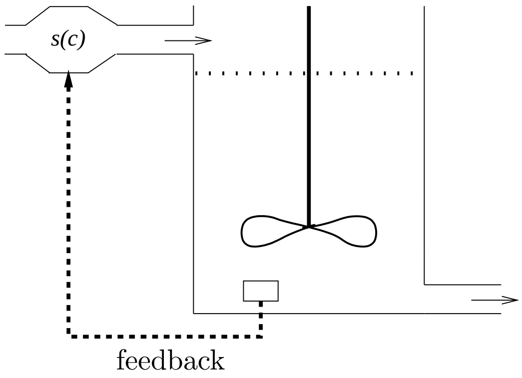

Figure 10.2: a mixing tank with a feedback control

With this control law, the differential equation (10.1) now becomes

$$c' = -pc + ps(c), \tag{10.2}$$

where the inflow concentration $s(c)$ depends on c.

A simple example of a feedback control law is the following one:

$$s = s(c) = \begin{cases} 0 & \text{if } c > k, \\ k & \text{if } c = k, \\ z & \text{if } c < k, \end{cases} \tag{10.3}$$

where z is some convenient value greater than k. The corresponding solution can be obtained as follows.

Case 1: If $c_0 = k$, then $s(c) = k$, and the solution is $c(t) = k$ for $t \geq 0$. Hence the desired concentration is reached at $t^* = 0$.

Case 2: If $c_0 > k$, then $s(c) = 0$ and the solution of Eq. (10.2) is

$$c(t) = c_0 \mathrm{e}^{-pt}.$$

The desired concentration will be attained at $t = t^*$, where $c(t^*) = k$. Hence $t^* = \frac{1}{p} \ln \left(\frac{c_0}{k} \right)$. At this time, the definition of the function s changes, so the solution continues with the solution obtained in Case 1, so the solution remains constant k for $t \geq t^*$.

Case 3: If $c_0 < k$, then $s(c) = z$, and from Eq. (10.2) the solution is

$$c(t) = z + (c_0 - z)\,\mathrm{e}^{-pt},$$

where $z > k > c_0$. The desired concentration will be obtained when $c(t^*) = k$, and this gives

$$t^* = \frac{1}{p} \ln \left(\frac{z - c_0}{z - k} \right).$$

But for $t \geq t^*$ the solution remains constant k by Case 1.

So the problem is solved from theoretical point of view, the solution always reaches the desired concentration in finite time and after that it remains constant. In the practice, the following modified versions described in the next section are more realistic.

10.2 Models with time delay

(i) Time is needed to sense information and to react to it, so we assume the inflow rate at time t depends on the result of the measurement at an earlier time $t-\tau$, where τ is a positive time delay.

So the governing equation is a delay differential equation

$$c'(t) = -pc(t) + ps(c(t-\tau)), \qquad t \geq 0, \tag{10.1}$$

with the initial condition

$$c(t) = c_0, \qquad -\tau \leq t \leq 0.$$

(ii) In the above models, the measurement is continuous which can be technically complicated and also expensive. It is more reasonable to assume that we measure the concentration in the tank only at discrete time moments. For simplicity, we take the following discrete moments

$$0, \quad h, \quad 2h, \quad ..., \quad nh, \quad ...$$

where $h > 0$ is the sampling period. The feedback control term in this case can be written as

$$ps\left(c\left(\left[\frac{t}{h}\right]h - \ell h\right)\right), \qquad t \geq 0,$$

where $s(c)$ is a given function, ℓ is a fixed positive integer describing the delay and $[\cdot]$ denotes the greatest integer part function. For instance, $[1.5] = 1$ and $[-0.25] = -1$.

So the governing equation is

$$c'(t) = -pc(t) + ps\left(c\left(\left[\frac{t}{h}\right]h - \ell h\right)\right), \qquad t \geq 0, \tag{10.2}$$

with the initial condition

$$c(t) = c_0, \qquad -\ell h \leq t \leq 0. \tag{10.3}$$

Eq. (10.2) is called an *equation with piecewise constant argument.* This is an example of a so-called *hybrid* delay differential equation, since it contains both continuous and discrete arguments. Its solution leads to the solution of some related discrete difference equation. Namely, multiplying both sides of Eq. (10.2) by e^{pt} gives

$$\mathrm{e}^{pt}c'(t) + p\mathrm{e}^{pt}c(t) = p\mathrm{e}^{pt}s\left(c\left(\left[\frac{t}{h}\right]h - \ell h\right)\right),$$

and integrating both sides of this equation from nh to t where $t \in [nh, (n+1)h)$ we get

$$\mathrm{e}^{pt}c(t) - \mathrm{e}^{pnh}c(nh) = \int_{nh}^{t} \mathrm{e}^{pu}ps\Big(c\Big(\Big[\frac{u}{h}\Big]h - \ell h\Big)\Big)\mathrm{d}u,$$

hence

$$c(t) = \mathrm{e}^{-p(t-nh)}c(nh) + \mathrm{e}^{-pt}\int_{nh}^{t} \mathrm{e}^{pu}ps\Big(c\Big(\Big[\frac{u}{h}\Big]h - \ell h\Big)\Big)\mathrm{d}u, \quad nh \leq t < (n+1)h.$$

But for $nh \leq u < t \leq (n+1)h$, we know that the argument is constant:

$$\left[\frac{u}{h}\right]h - \ell h = (n-\ell)h,$$

hence easy calculation shows

$$c(t) = \mathrm{e}^{-p(t-nh)}c(nh) + \mathrm{e}^{-pt}(\mathrm{e}^{pt} - \mathrm{e}^{pnh})s(c((n-\ell)h)), \quad nh \le t < (n+1)h. \tag{10.4}$$

Taking the limit $t \to (n+1)h-$, we arrive to the difference equation

$$c((n+1)h) = \mathrm{e}^{-ph}c(nh) + (1 - \mathrm{e}^{-ph})s(c((n-\ell)h)), \qquad n \ge 0. \tag{10.5}$$

Using the initial condition (10.3) it can be solved for $c(nh)$, $n \in \mathbb{N}$. In between the mesh points, the solution can be determined using formula (10.4).

Example 10.1 Consider the hybrid delay model (10.2), where instead of (10.3), we define the function s by

$$s = s(c) = \begin{cases} 0 & \text{if } c > k + \varepsilon, \\ k & \text{if } k - \varepsilon \le c \le k + \varepsilon, \\ z & \text{if } c < k - \varepsilon, \end{cases}$$

where $\varepsilon > 0$ is a small number. This modification of the definition of s is reasonable, since the discrete sequence $c(nh)$ may not take the value k.

We use the parameter values $p = 0.3$, $k = 3$, $\ell = 2$, $\varepsilon = 0.2$ and $h = 0.5$. In Figure 10.3, we generated the solution of (10.5) starting from the initial concentration $c_0 = 0.4$, and in Figure 10.4, the solution corresponding to $c_0 = 5$. In both cases the solutions approach $k = 3$.

The above two particular examples show cases when the solution approaches the limit k. But this is true only for particular parameter values. The analysis of this model is beyond the scope of these lecture notes. We comment that, e.g., for larger delay $\ell = 3$, the solution becomes periodic as time increases, see Figure 10.5. □

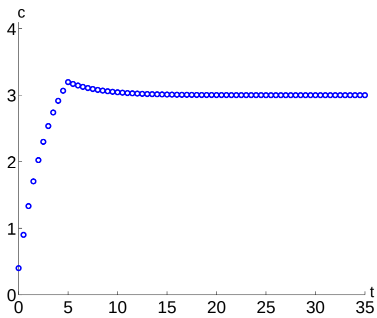

Figure 10.3: $c_0 = 0.4$, $\ell = 2$

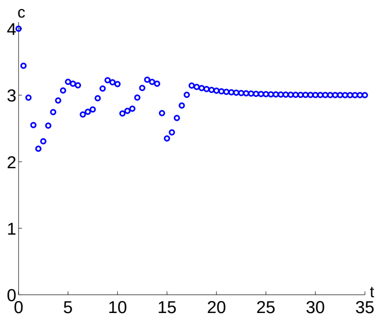

Figure 10.4: $c_0 = 4$, $\ell = 2$

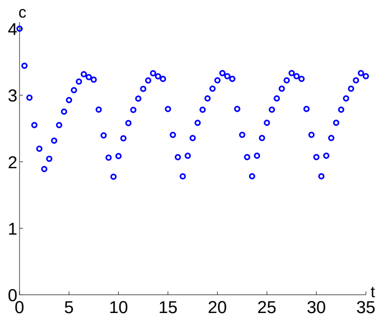

Figure 10.5: $c_0 = 4$, $\ell = 3$

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