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# Linear Algebra
## *Workbook*

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BEFEKTETÉS A JÖVŐBE

# Introduction

This material is made for the Linear Algebra course VEMKMA1143G at the University of Pannonia. The classes usually include students from the Faculty of Engineering and the Faculty of Economics. Some of them study for their BSc degree, some of them for Masters. This variety of background and motivation is a really hard challenge for the course instructor. Therefore, we collected the material in a very compact form. At the lectures and tutorials we try to cover parts of this note with extended material and extra explanations. However, these notes try to serve as the spine of the course material.

Linear Algebra is the branch of Mathematics concerning linear equations, linear maps and their representations through matrices and vector spaces. It also has practical applications in our modern daily life in various industries. We collected the fundamental material in eight Sections. The course builds on previous mathematical knowledge in Euclidean Geometry, Elementary Algebra, a little bit of Group Theory comes very useful as well.

The Sections are short and include only the most fundamental results and facts, usually without proofs. Also each section contains some examples with full solutions. The student readers should understand these explanations. After that, we hope the reader can solve the exercises at the end of each section.

The workbook finishes with the solution of the exercises.

Veszprém, December 2018.

János Barát
Associate Professor

# 1 Systems of linear equations

## 1.1 Two equations and two unknowns

In high school, we learnt how to solve a system of two linear equations in two unknowns. The main idea behind our method was the following. If we add two equations, that is, we add the left-hand sides and the right-hand sides of two equations, then we get another valid equation. Let us see the following

$$2x + 3y = 11 \tag{1.1}$$
$$4x + 5y = 6 \tag{1.2}$$

Here, we add (-2) times equation (1.1) to equation (1.2) to get

$$-y = -16 \tag{1.3}$$

Now we can add 3 times equation (1.3) to equation (1.1) to get

$$2x = -37 \tag{1.4}$$

Hence the final solution is $x = -\frac{37}{2}$ and $y = 16$.

What we were doing here, can be generalised to any system of linear equations in $m$ equations in $n$ unknowns. Before doing so, we recall that the equations of the form $ax + by = c$ are called *linear*, since the solution set of such equations correspond to *lines* in a Cartesian coordinate system. What we mainly expect is that two lines in the plane have a unique intersection point, just as it happened in our first example. However, there are two other possibilities:

**Example 1.1.** *Consider the system*

$$2x + 3y = 6 \tag{1.5}$$
$$4x + 6y = 12 \tag{1.6}$$

*It is apparent that the two equations carry the same information and the pair* $(x, \frac{6-2x}{3})$ *is a solution of the system for any real number* $x$*. Therefore, the system has an infinite number of solutions. Geometrically, the two lines corresponding to the two equations coincide in this case.*

**Example 1.2.** *Consider the system*

$$2x + 3y = 6 \tag{1.7}$$
$$4x + 6y = 11 \tag{1.8}$$

*Now multiply the first equation by* 2*, and we see that the two equations are contradictory. The system has no solution. Geometrically, the two lines corresponding to the two equations are parallel.*

In what follows, we will see that this trichotomy applies to the general case (that is, when there are $m$ equations and $n$ unknowns).

## 1.2 Gauss-Jordan elimination

In this section, we describe a general method for finding all solutions to a system of $m$ linear equations in $n$ unknowns. First we look at the case $m = 3, n = 3$. We use the notation $x_1, x_2, x_3$ for the variables.

**Example 1.3.** *Solve the following system:*

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 18 \\ 4x_1 + 5x_2 + 6x_3 &= 24 \\ 3x_1 + x_2 - 2x_3 &= 4 \end{aligned} \tag{1.9}$$

**Solution**: Our method will be to simplify the equations as we did in the previous section. We begin by dividing the first equation by 2. This gives us

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 4x_1 + 5x_2 + 6x_3 &= 24 \\ 3x_1 + x_2 - 2x_3 &= 4 \end{aligned} \tag{1.10}$$

As we saw in the previous section, adding two equations together leads to a third, valid equation. This equation may replace either of the two equations used to obtain it in the system. We begin the simplification of the system by multiplying the first equation by $(-4)$ and adding it to the second equation. This leads to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ -3x_2 - 6x_3 &= -12 \\ 3x_1 + x_2 - 2x_3 &= 4 \end{aligned} \tag{1.11}$$

Now we multiply the first equation by $(-3)$ and add it to the third equation.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ -3x_2 - 6x_3 &= -12 \\ -5x_2 - 11x_3 &= -23 \end{aligned} \tag{1.12}$$

Note that in system (1.12) the variable $x_1$ has been eliminated from the second and third equation. Next we divide the second equation by $-3$.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ x_2 + 2x_3 &= 4 \\ -5x_2 - 11x_3 &= -23 \end{aligned} \tag{1.13}$$

We multiply the second equation by $(-2)$ and add it to the first and then multiply the second equation by 5 and add it to the third:

$$\begin{array}{ccccc} x_1 & & -x_3 & = & 1 \\ & x_2 & +2x_3 & = & 4 \\ & & -x_3 & = & -3 \end{array}$$

Now we simply multiply the third equation by $(-1)$.

$$\begin{array}{ccccc} x_1 & & -x_3 & = & 1 \\ & x_2 & +2x_3 & = & 4 \\ & & x_3 & = & 3 \end{array}$$

Finally, we add the third equation to the first and then multiply the third equation by $(-2)$ and add it to the second. We obtain a system that is equivalent to (1.9):

$$\begin{array}{ccccc} x_1 & & & = & 4 \\ & x_2 & & = & -2 \\ & & x_3 & = & 3 \end{array} \tag{1.14}$$

This is the unique solution to the system. The method we used here is the **Gauss-Jordan elimination**.

We introduce a notation that makes our life easier. A **matrix** is a rectangular array of numbers. We will study matrices in detail in Section 4. For instance, the coefficients of the variables in system (1.9) can be written as the entries of a matrix $A$, called the coefficient matrix of the system:

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{pmatrix}.$$

We will repeatedly use three simple steps to achieve the **reduced row echelon form** (as in (1.2)) from the coefficient matrix. This was illustrated by an example in the beginning of this section. The three possible steps are the

| | **Elementary row operations** |
|---|---|
| **i** | Multiply a row by a nonzero number. |
| **ii** | Add a multiple of one row to another row. |
| **iii** | Interchange two rows. |

**Example 1.4.** The following series of matrices show a typical Gauss-Jordan elimination process.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first arrow hides the following two elementary row operations: We subtract row 1 from row 2 and we subtract 3 times row 1 from row 3. The second arrow means the following: We subtract row 2 from row 3. The third arrow corresponds to the following: We multiply row 2 by -1/2. We can also say that we divided by -2. After that we subtract 3 times row 2 from row 1.

At the end, we read off the following: $x_1 - 2x_3 = -3$ and $x_2 + x_3 = 4$. That is, we have the freedom to set the value of $x_3$ (free variable), and after that $x_1$ and $x_2$ are uniquely determined. However, we have an infinite number of choices for $x_3$, there are infinitely many solutions to the system.

For instance, setting $x_3 = 4$ gives us the solution: $x_1 = 5$, $x_2 = 0$, $x_3 = 4$. We can check the equations:
$5 + 3 \times 0 + 4 = 9$
$5 + 0 - 4 = 1$ and
$3 \times 5 + 11 \times 0 + 5 \times 4 = 35$. □

**Remark 1.5.** *If there are more variables than equations, then there will always be some free variables.*

**Remark 1.6.** *If we get a row, where all instances are 0, but the right-hand side is non-zero, then we got a contradiction. In that case, there is no solution to the system.*

## Excercises

Solve the following systems using the Gauss-Jordan elimination.

1. $$\begin{array}{rcrcrcr} 3x_1 & + & 3x_2 & + & x_3 & = & 5 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 1 \\ 2x_1 & + & x_2 & + & 3x_3 & = & 11 \end{array}$$

2. $$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 5x_3 & + & 7x_4 & = & 12 \\ 3x_1 & + & 5x_2 & + & 7x_3 & + & x_4 & = & 0 \\ 5x_1 & + & 7x_2 & + & x_3 & + & 3x_4 & = & 4 \\ 7x_1 & + & x_2 & + & 3x_3 & + & 5x_4 & = & 16 \end{array}$$

3. $$\begin{array}{rcrcrcr} 2x & + & y & - & z & = & 8 \\ -3x & - & y & + & 2z & = & -11 \\ -2x & + & y & + & 2z & = & -3 \end{array}$$

4. $$\begin{array}{rcrcrcr} 2x & + & 4y & - & 6z & = & 18 \\ 4x & + & 5y & + & 6z & = & 24 \\ 2x & + & 7y & + & 12z & = & 40 \end{array}$$

5. $$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 7 \\ 3x_1 & + & 6x_2 & - & 3x_3 & + & 3x_4 & = & 21 \end{array}$$

# 2 Vectors

## 2.1 Addition and scalar multiples of vectors

We define an $n$-component row vector as an ordered set of $n$ numbers written as $(x_1, x_2, \ldots, x_n)$. Here $x_i \in \mathbb{R}$ for all $i$. Similarly an $n$-component column vector is an ordered set of $n$ numbers written as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Here also $x_i$ is called the $i$th coordinate. If a vector has $k$ coordinates, then we call it a $k$-vector. The natural notation is used for the all-zero vector (of any size): $\mathbf{0} = (0, 0, \ldots 0)$ or $\underline{0}$. The word *ordered* is important in the definition of a vector. The two vectors $(1, 2)$ and $(2, 1)$ are different. In this text, we denote the vectors by boldface lower-case letters as: $\mathbf{v}, \mathbf{w}, \mathbf{t}$ etc. or underlined lower-case letters $\underline{v}, \underline{u}, \underline{x}$ etc. Two vectors $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_l)$ are equal if and only if they have the same number of components and they are all equal. That is $k = l$ and $a_i = b_i$ for all $i$.

**Addition**. Let $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$ be two vectors, then their sum is $\mathbf{a} + \mathbf{b} = (a_1 + b_1, \ldots, a_k + b_k)$.

**Example**. $(3, 2, 4) + (1, -1, -2) = (4, 1, 2)$.

**Multiplication by a scalar**. Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a vector and $\alpha \in \mathbb{R}$. Then the product $\alpha\mathbf{a}$ is given by $(\alpha a_1, \ldots, \alpha a_k)$.

We can combine the two operations.

**Example 2.1.** *Let* $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$. *Calculate* $2\mathbf{a} + 3\mathbf{b}$.

**Solution:** $2\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 2 \cdot 1 - 3 \cdot 1 \\ 2 \cdot 2 + 3 \cdot 3 \\ 2 \cdot 3 + 3 \cdot 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 13 \\ 15 \end{pmatrix}$.

We can prove a number of facts regarding these operations:

**Theorem 2.2.** *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be vectors of the same size and* $\alpha, \beta$ *be scalars. Then the following hold:*

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. *Commutative law.*

$\mathbf{a} + \mathbf{0} = \mathbf{a}$.

$0\mathbf{a} = \mathbf{0}$.

$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$. *Associative law.*

$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$.

$(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$.

$(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$.

Another notion we learnt in high school, is the length of a vector in two dimensions. That arises as a direct application of the Pythagorean theorem. Now if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the **length** is $\sqrt{x_1^2 + \cdots + x_n^2}$ and denoted by $|\mathbf{x}|$.

# Excercises

1. Let $\mathbf{a} = (8, 6, 8, 5)$ and $\mathbf{b} = (-5, -9 - 1, 3)$. Calculate $-4\mathbf{a} + 7\mathbf{b}$ and $-7(\mathbf{a} + \mathbf{b})$.
2. Let $\mathbf{a} = (4, 4 - 7)$, $\mathbf{b} = (-8, 1, 4)$ and $\mathbf{c} = (3, 4 - 5)$. Calculate $\mathbf{a} + \mathbf{b} + 4\mathbf{c}$.
3. Let $\mathbf{a} = (-3, -4, 5, -9)$. What is the length of $\mathbf{a}$?

## 2.2 Product of two vectors

**Scalar product.** Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two vectors of the same size. The scalar product of $\mathbf{a}$ and $\mathbf{b}$, denoted as $\mathbf{a} \cdot \mathbf{b}$ is given by $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n$. Note that the result of the scalar product is a number. There are alternate names for this product: inner product, dot product. Sometimes the vectors can be column vectors or one of each type. The important thing is that they have the same number of components.

**Example 2.3.** *Let* $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$. *Let us calculate* $\mathbf{a} \cdot \mathbf{b}$.

**Solution:** $\mathbf{a} \cdot \mathbf{b} = (1)(-1) + (2)(3) + (3)(3) = -1 + 6 + 9 = 14.$ □

**Example 2.4.** *Let* $\mathbf{a} = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 3 \end{pmatrix}$ *and* $\mathbf{b} = (2, -2, 1, 0)$. *Calculate the dot product.*

**Solution:** $\mathbf{a} \cdot \mathbf{b} = (4)(2) + (2)(-2) + (1)(1) + (3)(0) = 8 - 4 + 1 + 0 = 5.$ □

**Theorem 2.5.** *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *be* $n$*-vectors and let* $\alpha$ *be a scalar. The following rules hold:*
$\mathbf{a} \cdot \mathbf{0} = 0$.
$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. *Commutative law.*
$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$. *Distributive law.*
$(\alpha \mathbf{a}) \cdot \mathbf{b} = \alpha(\mathbf{a} \cdot \mathbf{b})$.

# Excercises

1. Calculate the scalar product of the two vectors:
   $(-2, -1, -5, 2)$ and $(-4, -1, -4, 4)$. $(3, 8, 0)$ and $(-8, 0, -7)$. $(8, 1, -3)$ and $(3, -4, 7)$.
2. Calculate the scalar product of the two vectors:
   $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} y \\ z \\ x \end{pmatrix}$.
3. Let $\mathbf{a}$ be a $k$-vector. Show that $\mathbf{a} \cdot \mathbf{a} \geq 0$.
4. Perform the indicated computations with $\mathbf{a} = \begin{pmatrix} 7 \\ -3 \\ -8 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -2 \\ 5 \end{pmatrix}$.

   $2\mathbf{a} \cdot 3\mathbf{b}$, $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$, $(2\mathbf{b}) \cdot (3\mathbf{c} - 5\mathbf{a})$, $(\mathbf{a} - \mathbf{c}) \cdot (3\mathbf{b} - 4\mathbf{a})$

**Vector product.** One suspects that there might be a meaningful way to associate a vector to a pair of vectors. The notion of a *vector product* exists in three dimensions. We postpone its definition to Section 5 and make use of it in Section 6.

## 2.3 Vectors in space

The most typical usage of vectors happens in three dimensions. In real space, where we live. It is natural, that we can describe the exact place of a point-like object using three coordinates. For that, we usually use a Cartesian coordinate system with three axis. When we are given only two vectors, they always lie in a plane, although they have three coordinates. Therefore, we can use our knowledge of Euclidean Geometry. For instance the law of cosines is a generalisation of Pythagoras Theorem: In a triangle with sidelength $a, b$ and $c$, where the angle opposite to the side of length $c$ is $\gamma$, the following holds:
$c^2 = a^2 + b^2 - 2ab\cos\gamma$.

**Theorem 2.6.** *In the three-dimensional space, the dot product of* $\mathbf{a}$ *and* $\mathbf{b}$ *can be also calculated as follows:* $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\gamma$*, where* $\gamma$ *is the angle between the two vectors.*

**Example 2.7.** *Let* $\underline{a} = (1, 0, 2)$ *and* $\underline{b} = (2, 5, 3)$ *be two vectors in space.*

*a. Draw the two vectors in a 3-dimensional Cartesian coordinate system.*
*b. Calculate the vector* $2\underline{a} + 7\underline{b}$*.*
*c. What is the length of vectors* $\underline{a}$ *and* $\underline{b}$*?*
*d. Determine the angle between* $\underline{a}$ *and* $\underline{b}$*?*
*e. Give the opposite of* $\underline{a}$*, a vector parallel to* $\underline{a}$ *and one perpendicular to* $\underline{a}$*.*
*f. What is the unit vector parallel to* $\underline{a}$*?*
*g. Calculate the vectors of length* 3 *and length* 1/2 *parallel to* $\underline{a}$*.*

**Solution:**

b. $2\underline{a} + 7\underline{b} = 2 \cdot (1, 0, 2) + 7 \cdot (2, 5, 3) = (2, 0, 4) + (14, 35, 21) = (16, 40, 24)$.

c. The length of $\underline{a}$: $|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}$.

The length of $\underline{b}$: $|\underline{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38}$.

d. Let us denote by $\gamma$ the angle between $\underline{a}$ and $\underline{b}$. Now $\cos\gamma = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \dfrac{1 \cdot 2 + 0 \cdot 5 + 2 \cdot 3}{\sqrt{5}\sqrt{38}} = \dfrac{8}{\sqrt{190}}$. Hence $\gamma = \arccos \frac{8}{\sqrt{190}}$.

e. The opposite of $\underline{a}$ is $-\underline{a} = (-1, 0, -2)$.

Any vector parallel to $\underline{a}$ is a scalar multiple of $\underline{a}$. For instance, $3\underline{a} = (3, 0, 6)$ or $-0.7\underline{a} = (-0.7, 0, -1.4)$.

One way of determining the vectors perpendicular to $\underline{a}$ is the following: Any such vector $\underline{x} = (x_1, x_2, x_3)$ has inner product 0 with $\underline{a}$. Therefore, the following equation holds:

$$1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 = 0.$$

Since $x_2$ disappears, we might choose $x_2$ and one other coordinate of $\underline{x}$ arbitrarily and then calculate the third one using the above equation. Let $x_1 = 5$ and $x_2 = 10$. Now $1 \cdot 5 + 0 \cdot 10 + 2 \cdot x_3 = 0$. That is, $x_3 = -\frac{5}{2}$ and $(5, 10, -\frac{5}{2})$ is a vector perpendicular to $\underline{a}$.

f. The unit vector in the same direction as $\underline{a}$ is the following: $u = \frac{\underline{a}}{|\underline{a}|} = \frac{1}{\sqrt{5}}(1, 0, 2) = (\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}})$

g. We calculate this, using the previous answer. The vector of length 3 parallel to $\underline{a}$ is $3 \cdot u = (\frac{3}{\sqrt{5}}, 0, \frac{6}{\sqrt{5}})$. Similarly the vector of length 1/2 parallel to $\underline{a}$ is $1/2 \cdot u = (\frac{1}{2\sqrt{5}}, 0, \frac{1}{\sqrt{5}})$. □

**Example 2.8.** *Let* $\underline{v} = (3, -1, 2)$ *and* $\underline{a} = (1, 1, -2)$*.*

*a/ Find the projection of* $\underline{v}$ *on* $\underline{a}$*.*
*b/ Split vector* $\underline{v}$ *into components parallel and perpendicular to* $\underline{a}$*.*

**Solution:**

a/ We have to use the following formula: $\text{proj}_{\underline{a}}\underline{v} = \dfrac{\underline{v} \cdot \underline{a}}{|\underline{a}|^2}\underline{a}$.

Now $|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$ and $\underline{v} \cdot \underline{a} = 3 \cdot 1 - 1 \cdot 1 - 2 \cdot 2 = -2$.

Therefore, $\text{proj}_{\underline{a}}\underline{v} = -\frac{1}{3}\underline{a} = (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3})$.

b/ By definition, the component parallel to $\underline{a}$ is $\text{proj}_{\underline{a}}\underline{v}$, which we determined in the previous part. On the other hand, the component perpendicular to $\underline{a}$ is $\underline{v} - \text{proj}_{\underline{a}}\underline{v} = (3, -1, 2) - (-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}) = (\frac{10}{3}, -\frac{2}{3}, \frac{4}{3})$. □

# Excercises

1. Let $\underline{u} = (2, 3, -1)$ and $\underline{v} = (0, -1, 4)$ be two vectors in space.
   (a) Draw the two vectors in a 3-dimensional Cartesian coordinate system.
   (b) Calculate the vector $2\underline{v} - 3\underline{u}$.
   (c) What is the length of vectors $\underline{u}$ and $\underline{v}$?
   (d) Determine the angle between $\underline{u}$ and $\underline{v}$?

(e) Give the opposite of $\underline{v}$, a vector parallel to $\underline{v}$ and one perpendicular to $\underline{v}$.

(f) What is the unit vector parallel to $\underline{v}$?

(g) Calculate the vectors of length 4 and length 1/3 parallel to $\underline{v}$.

2. Let $\underline{v} = (4, 7, 9)$ and $\underline{a} = (2, -1, 3)$.

(a) Find the projection of $\underline{v}$ on $\underline{a}$.

(b) Split vector $\underline{v}$ into components parallel and perpendicular to $\underline{a}$.

# 3 Linear independence, dimension in vector spaces

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors that have the same number of components, then any vector of the form $\mu\mathbf{a}+\lambda\mathbf{b}$ is a **linear combination** of $\mathbf{a}$ and $\mathbf{b}$, where $\mu$ and $\lambda$ are real numbers. Notice that $\mu$ and $\lambda$ can be zero as well. Therefore, all linear combinations of two vectors usually constitute a plane. The set of all linear combinations is also called the **span**.

**Example 3.1.** *Let* $\mathbf{a} = (-2, 3-4)$ *and* $\mathbf{b} = (1, 1, 5)$. *Can one write* $\mathbf{x} = (5, 3, 1)$ *or* $\mathbf{y} = (-5, 5, -13)$ *as a linear combination of* $\mathbf{a}$ *and* $\mathbf{b}$*?*

**Solution:** We write a linear combination as follows: $\alpha\mathbf{a} + \beta\mathbf{b}$. This expression gives us the following equations for the coordinates of $\mathbf{x} = (5, 3, 1)$:

$$\begin{aligned} -2\alpha + \quad \beta &= 5 \\ 3\alpha + \quad \beta &= 3 \\ -4\alpha + 5\beta &= 1 \end{aligned}$$

Solving the first two equations for $\alpha$ and $\beta$, we get $\alpha = -0.4$ and $\beta = 4.2$. However, these values give a contradiction in the third equation. Therefore $\mathbf{x}$ does not belong to the span of $\mathbf{a}$ and $\mathbf{b}$.

For $\mathbf{y} = (-5, 5, -13)$, we get the following system:

$$\begin{aligned} -2\alpha + \quad \beta &= -5 \\ 3\alpha + \quad \beta &= 5 \\ -4\alpha + 5\beta &= -13 \end{aligned}$$

Solving this, we get $\alpha = 2$ and $\beta = -1$. Therefore $\mathbf{y} = 2\mathbf{a} - 1\mathbf{b}$. □

We can generalise linear combination to any number of vectors. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ be a set of vectors that have the same number of components. If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are real numbers, then $\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \ldots + \alpha_k\mathbf{v_k}$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$. All linear combinations of a set of vectors $S$ is the **span** of $S$. Another very important concept is **linear independence**. We say that the set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is linearly independent, if the only solution to the equation $\alpha_1\mathbf{v_1}+\alpha_2\mathbf{v_2}+\ldots+\alpha_k\mathbf{v_k} = \mathbf{0}$ is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Equivalently, we could define **linear dependence**, and sometimes this is useful. We say that a non-zero vector $\mathbf{x}$ is linearly dependent on the vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$, if the equation $\mathbf{x} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \ldots + \alpha_k\mathbf{v_k}$ has a solution. This automatically means that at least one of the coefficients is non-zero.

If we have an arbitrary set of vectors, then the **rank** of the set is the maximum number of linearly independent vectors in the set. A set $H$ of vectors form a *generating set* of $V$, if every vector $v \in V$ can be written as a linear combination of some vectors in $H$. That is, $H$ spans $V$. As we will se, our goal is to have larger and larger independent sets of vectors and similarly smaller and smaller sets of vectors generating the same set.

We are working with sets of vectors. Some of these collection of vectors form a closed compact structure. We have already experienced that certain calculations with 2- or 3-component vectors is comfortable. We formalize this in the following fundamental notion.

**Definition 3.2.** *A* **real vector space** $V$ *is a set of vectors, together with two operations: addition and scalar multiplication, where the scalars are real numbers. The set of vectors must satisfy certain nice properties:*

*closed under addition and scalar multiplication*
*addition is associative, commutative*
*there is a zero vector, additive identity*
*scalar multiplication is associative*
*there is a multiplicative identity*
*there are two distributive laws*

**Example 3.3.** *Let* $V = \{(1)\}$. *That is a single vector with one component. Is it a vector space?*

**Solution:** This is not a vector space, since it is not closed under addition: $(1) + (1) = (2) \notin V$. □

**Example 3.4.** *Let* $V = \{(0, 0)\}$. *That is a single vector with two coordinates each of them being 0.*

**Solution:** This is a vector space! We can check all properties. For instance, $(0, 0) + (0, 0) = (0, 0)$. Also for any real number $\alpha$, we get $\alpha(0, 0) = (0, 0)$. There is a multiplicative identity, for instance 2. □

**Example 3.5.** *The set of points in $\mathbb{R}^2$ that lie on a line passing through the origin constitutes a vector space.*

**Solution:** We can check all properties. For instance, closed under scalar multiplication, since the scalar multiples of a vector are parallel to the original one. For the existance of the additive identity, it is important that the line goes through the origin. Points of other lines do not form a vector space. □

Let $H = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ be a set of $n$-vectors. If $H$ is linearly independent and spans all vectors of a vector space $V$, then $H$ is a **basis** of $V$. In most cases, we work in the vector space $\mathbb{R}^n$. That is, the set of all real vectors with $n$ components. This particular vector space has the following **standard basis**: $(1, 0, \dots, 0)$,$(0, 1, 0, \dots, 0)$,..., $(0, 0, \dots, 0, 1)$. Here each vector contains $n-1$ zeroes and one 1. We denote the elements of the standard basis by $\mathbf{e_i}$, where the index $i$ shows the position of the 1. For instance $\mathbf{v} = (a, b, c) \in \mathbb{R}^3$ can be written as the linear combination of $\mathbf{e_1}, \mathbf{e_2}$ and $\mathbf{e_3}$ as follows: $\mathbf{v} = a \cdot \mathbf{e_1} + b \cdot \mathbf{e_2} + c \cdot \mathbf{e_3}$.

**Proposition 3.6.** *If $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ is a basis of $V$, then every vector $\mathbf{v} \in V$ can be uniquely written as $\alpha_1\mathbf{v_1} + \dots + \alpha_k\mathbf{v_k}$, where $\alpha_i \in \mathbb{R}$.*

If the vector space $V$ has a finite basis, then the number of elements in the basis is the **dimension** of $V$. There exist vector spaces such that the number of elements in the basis is infinite. In that case the dimension is $\infty$. The vector space consisting of the sole zero vector has dimension 0.

**Lemma 3.7** (Steinitz exchange lemma)**.** *If $L = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}\}$ is a basis of the vector space $V$ and $G = \{\mathbf{g_1}, \mathbf{g_2}, \dots, \mathbf{g_n}\}$ spans $V$, then $k \le n$ and possibly after reordering the $\mathbf{g_i}$, the set $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_k}, \mathbf{g_{k+1}}, \dots, \mathbf{g_n}\}$ spans $V$.*

**Proof idea:** We use induction. The inductive step is the following: for any vector $\mathbf{v} \in L$, there exists a vector $\mathbf{g} \in G$ such that $G \setminus \mathbf{g} \cup \mathbf{v}$ spans $V$. □

**Corollary 3.8.**
*(i) If $L$ is a set of linearly independent vectors in a vector space $V$ and $G$ spans $V$, then for any $\mathbf{v} \in L$, there exists a $\mathbf{g} \in G$ such that $L \setminus \mathbf{v} \cup \mathbf{g}$ is linearly independent. (elementary basis exchange)*
*(ii) If $L$ is a set of linearly independent vectors, $B$ a basis and $G$ spans the vector space $V$, then $|L| \le |B| \le |G|$.*
*(iii) Any two basis of a fixed vector space $V$ must have the same number of elements. This number is the* **dimension** *of $V$.*
*(iv) Any $n$ independent vectors of a vector space of dimension $n$ form a basis.*

Let $B = \{\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_n}\}$ be a basis, and let $\mathbf{v}$ be an arbitrary vector in $V$. We know that $\mathbf{v} = \alpha_1\mathbf{b_1} + \dots + \alpha_n\mathbf{b_n}$ for some $\alpha_1, \dots, \alpha_n$, the coefficients (or coordinates) with respect to the basis $B$. After the previous lemma and its corollaries, we face the following. What happens if we change a basis $B_1$ to another basis $B_2$. If we know the coefficients with respect to basis $B_1$, how can we calculate the coefficients with respect to the new basis $B_2$? This is given by an algorithm, the basis exchange process. Each step is an elementary basis exchange, whose calculations coincide with that of an elimination step of the Gauss-Jordan elimination. We illustrate this in the following example.

**Example 3.9.** *Let $\mathbf{a_1} = (2, 3, 1, 4)$, $\mathbf{a_2} = (1, 1, 2, 2)$, $\mathbf{a_3} = (0, 0, 1, 1)$ and $\mathbf{a_4} = (3, -1, -2, -4)$. Do $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}$ form a basis in $\mathbb{R}^4$? If they do, then determine the coefficients of $\mathbf{v} = (9, 1, 2, -2)$ with respect to this basis.*

**Solution:** We start with the canonical basis of $\mathbb{R}^4$. In each step, we try to include a new vector from our set to the basis using the exchange lemma. The starting table looks like this:

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{v}$ |
|---|---|---|---|---|---|
| $\mathbf{e_1}$ | 2 | 1 | 0 | 3 | 9 |
| $\mathbf{e_2}$ | 3 | 1 | 0 | $-1$ | 1 |
| $\mathbf{e_3}$ | 1 | 2 | 1 | $-2$ | 2 |
| $\mathbf{e_4}$ | 4 | 2 | 1 | $-4$ | $-2$ |

In each step, we select a non-zero entry, preferably a 1, and perform an elementary exchange. We get the following series of tables.

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{v}$ |
|---|---|---|---|---|---|
| $\mathbf{a_2}$ | 2 | 1 | 0 | 3 | 9 |
| $\mathbf{e_2}$ | 1 | 0 | 0 | −4 | −8 |
| $\mathbf{e_3}$ | −3 | 0 | 1 | −8 | −16 |
| $\mathbf{e_4}$ | 0 | 0 | 1 | −10 | −20 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{v}$ |
|---|---|---|---|---|---|
| $\mathbf{a_2}$ | 0 | 1 | 0 | 11 | 25 |
| $\mathbf{a_1}$ | 1 | 0 | 0 | −4 | −8 |
| $\mathbf{e_3}$ | 0 | 0 | 1 | −20 | −40 |
| $\mathbf{e_4}$ | 0 | 0 | 1 | −10 | −20 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{v}$ |
|---|---|---|---|---|---|
| $\mathbf{a_2}$ | 0 | 1 | 0 | 11 | 25 |
| $\mathbf{a_1}$ | 1 | 0 | 0 | −4 | −8 |
| $\mathbf{a_3}$ | 0 | 0 | 1 | −20 | −40 |
| $\mathbf{e_4}$ | 0 | 0 | 0 | 10 | 20 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{v}$ |
|---|---|---|---|---|---|
| $\mathbf{a_2}$ | 0 | 1 | 0 | 0 | 3 |
| $\mathbf{a_1}$ | 1 | 0 | 0 | 0 | 0 |
| $\mathbf{a_3}$ | 0 | 0 | 1 | 0 | 0 |
| $\mathbf{a_4}$ | 0 | 0 | 0 | 1 | 2 |

Since we were able to include all four vectors, the set $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}\}$ form a basis in $\mathbb{R}^4$. The last column of the table shows that $\mathbf{v} = 3\mathbf{a_2} + 2\mathbf{a_4}$. □

We can use the basis exchange process to various other purposes. For instance, we can determine the rank of a set of vectors, $S$ say. In each step, we try to include a vector of $S$. This process might stop before we empty $S$. If there are rows filled with 0, that cannot be used for further exchanges. We always need a non-zero entry below the vector, we plan to include. Therefore, if we can include at most $k$ vectors, then the rank is $k$.

**Example 3.10.** *Let* $\mathbf{a_1} = (1, 1, 2)$, $\mathbf{a_2} = (2, 1, 0)$, $\mathbf{a_3} = (0, 1, 1)$, $\mathbf{a_4} = (8, 5, 4)$. *We form two sets of vectors* $H_1 = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$, $H_2 = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_4}\}$. *Determine whether* $H_1$ *and* $H_2$ *is linearly independent.*

**Solution:** We include the vectors into the basis exchange table. We try to perform elementary basis exchanges to include the vectors of $H_1$ or $H_2$. We start with $\mathbf{a_1}$ and $\mathbf{a_2}$, and see whether the third exchange is possible.

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{e_1}$ | 1 | 2 | 0 | 8 |
| $\mathbf{e_2}$ | 1 | 1 | 1 | 5 |
| $\mathbf{e_3}$ | 2 | 0 | 1 | 4 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{e_1}$ | 0 | 1 | −1 | 3 |
| $\mathbf{a_1}$ | 1 | 1 | 1 | 5 |
| $\mathbf{e_3}$ | 0 | −2 | −1 | −6 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{a_2}$ | 0 | 1 | −1 | 3 |
| $\mathbf{a_1}$ | 1 | 0 | 2 | 2 |
| $\mathbf{e_3}$ | 0 | 0 | −3 | 0 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{a_2}$ | 0 | 1 | 0 | 3 |
| $\mathbf{a_1}$ | 1 | 0 | 0 | 2 |
| $\mathbf{a_3}$ | 0 | 0 | 1 | 0 |

As we can read it, $H_1$ is linearly independent. However, $\mathbf{a_4} = 2\mathbf{a_1} + 3\mathbf{a_2}$. Therefore, $H_2$ is a set of linearly dependent vectors. □

**Example 3.11.** *Let* $\mathbf{a_1} = (1, 0, 2)$, $\mathbf{a_2} = (2, 1, 5)$, $\mathbf{a_3} = (-1, -1, -3)$, $\mathbf{a_4} = (5, 2, 12)$, $\mathbf{a_5} = (4, 2, 10)$, *and let* $H = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}\}$.

*Determine the rank of* $H$.

*Are there two linearly independent vectors in* $H$*? And two linearly dependent ones?*

*Can we add a new vector to* $H$ *such that the rank increases?*

**Solution:** The rank is given by the maximum number of vectors that can be included in the basis during the basis transformation. We start with the following table and perform elementary basis exchanges:

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{a_5}$ |
|---|---|---|---|---|---|
| $\mathbf{e_1}$ | 1 | 2 | −1 | 5 | 4 |
| $\mathbf{e_2}$ | 0 | 1 | −1 | 2 | 2 |
| $\mathbf{e_3}$ | 2 | 5 | −3 | 12 | 10 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{a_5}$ |
|---|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | 2 | −1 | 5 | 4 |
| $\mathbf{e_2}$ | 0 | 1 | −1 | 2 | 2 |
| $\mathbf{e_3}$ | 0 | 1 | −1 | 2 | 2 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ | $\mathbf{a_5}$ |
|---|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | 0 | 1 | 1 | 0 |
| $\mathbf{a_2}$ | 0 | 1 | −1 | 2 | 2 |
| $\mathbf{e_3}$ | 0 | 0 | 0 | 0 | 0 |

We included two vectors of $H$, therefore the rank is 2.

Now $\{\mathbf{a_1}, \mathbf{a_2}\}$ is a linearly independent set and $\{\mathbf{a_2}, \mathbf{a_5}\}$ form a linearly dependent set.

Every vector that is not a linear combination of $\mathbf{a_1}$ and $\mathbf{a_2}$ increases the rank. One such vector is $(0,0,1)$ since $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{e_3}\}$ form a basis. □

In the typical vector space $\mathbb{R}^n$, there might be a set $V$ of vectors that is closed with respect to addition and scalar multiplication. That is, $V$ itself satisfies the properties of a vector space. In that case we call $V$ a subspace. We have already encountered this phenomenon in the Example 3.5. Actually, in any dimension, the multiples of a fixed non-zero vector form a 1-dimensional subspace. The dimension of a subspace $V$ is the number of elements in a basis of $V$. A vector space $V$ is the direct sum of its two subspaces $V_1$ and $V_2$ if and only if every vector $v$ of $V$ can be uniquely written as the sum of two vectors $v_1 \in V_1$ and $v_2 \in V_2$. Necessarily the intersection of $V_1$ and $V_2$ must be the zero vector of $V$. Also the dimension $n_1$ of $V_1$ and $n_2$ of $V_2$ must satisfy $n_1 + n_2 = n$, where $n$ is the dimension of $V$.

**Example 3.12.** *Determine the dimension of the subspaces below and give a basis for each of them.*

$V_1 = \{\lambda_1(1,-2,3) + \lambda_2(1,0,1) : \lambda_1, \lambda_2 \in \mathbb{R}\}$, $V_2 = \{\lambda(1,0,0) : \lambda \in \mathbb{R}\}$.

*Is it true that* $V_1 \oplus V_2 = \mathbb{R}^3$*? In case of the affirmative, split the vector* $\mathbf{x} = (4,-2,5)$ *to components in* $V_1$ *and* $V_2$*.*

**Solution:** The dimension of $V_1$ is 2, and $(1,-2,3), (1,0,1)$ is a basis. The dimension of $V_2$ is 1, and $(1,0,0)$ is a basis.

Now we have to check whether their union is independent. We include them in a basis exchange table and solve it by the standard method. We include $\mathbf{x}$ to determine the coefficients in the possible basis. Since $\mathbf{e_1} = \mathbf{b_3}$ we can make this exchange for free.

| basis | $\mathbf{b_1}$ | $\mathbf{b_2}$ | $\mathbf{b_3}$ | $\mathbf{x}$ |
|---|---|---|---|---|
| $\mathbf{b_3}$ | 1 | 1 | 1 | 4 |
| $\mathbf{e_2}$ | −2 | 0 | 0 | −2 |
| $\mathbf{e_3}$ | 3 | 1 | 0 | 5 |

$\longrightarrow$

| basis | $\mathbf{b_1}$ | $\mathbf{b_2}$ | $\mathbf{b_3}$ | $\mathbf{x}$ |
|---|---|---|---|---|
| $\mathbf{b_3}$ | −2 | 0 | 1 | −1 |
| $\mathbf{e_2}$ | −2 | 0 | 0 | −2 |
| $\mathbf{b_2}$ | 3 | 1 | 0 | 5 |

$\longrightarrow$

| basis | $\mathbf{b_1}$ | $\mathbf{b_2}$ | $\mathbf{b_3}$ | $\mathbf{x}$ |
|---|---|---|---|---|
| $\mathbf{b_3}$ | 0 | 0 | 1 | 1 |
| $\mathbf{b_1}$ | 1 | 0 | 0 | 1 |
| $\mathbf{b_2}$ | 0 | 1 | 0 | 2 |

We found that indeed $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ is a basis and $\mathbf{x} = \mathbf{b_1} + 2\mathbf{b_2} + \mathbf{b_3}$. Therefore the $V_1$-component is $(3,-2,5)$ and the $V_2$-component is $(1,0,0)$ □

## Excercises

1. Let $\mathbf{a} = (2,-3)$, $\mathbf{b} = (0,5)$. Can we get $\mathbf{c} = (-2,23)$ as a linear combination of $\mathbf{a}$ and $\mathbf{b}$?

2. Let $\mathbf{a} = (5,4,-2,3)$, $\mathbf{b} = (2,0,-1,5)$, $\mathbf{c} = (3,0,4,-6)$. Can we get $\mathbf{x} = (6,4,0,19)$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$?

3. Let $\mathbf{a} = (-1,2,0)$, $\mathbf{b} = (3,5,2)$, and $\mathbf{c} = (-2,1,4)$. Let $H = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. How can we get the zero vector of $\mathbb{R}^3$ from the vectors of $H$? Is $H$ linearly independent?

4. Let $\mathbf{a_1} = (1,3,2)$, $\mathbf{a_2} = (2,1,5)$, $\mathbf{a_3} = (3,4,2)$. Do $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ form a basis of $\mathbb{R}^3$? If yes, calculate the coordinates of $\mathbf{v} = (14,17,18)$ with respect to this basis.

5. Let $\mathbf{a} = (2,4,-8)$, $\mathbf{b} = (-5,-9,18)$, $\mathbf{c} = (7,2,-7)$. Let $H = \{\underline{a}, \underline{b}, \underline{c}\}$. How can we get the zero vector using the elements in $H$? Is $H$ linearly independent? Let $\mathbf{x} = (0,1,-12)$ and $\mathbf{y} = (2,2,2)$. Can we get $\mathbf{x}$ or $\mathbf{y}$ as a linear combination of $\mathbf{a}$ and $\mathbf{b}$?

6. Let $H_1 = \{(1,1,1), (1,1,0)\}$,
   $H_2 = \{(1,1,1), (1,1,0), (1,0,0)\}$ and
   $H_3 = \{(1,1,1), (1,1,0), (1,0,0), (0,1,1)\}$.
   Check the following features for each vector set: linear independence, basis, generating set.

7. Let $\mathbf{a_1} = (1,2,4)$, $\mathbf{a_2} = (-3,1,2)$, $\mathbf{a_3} = (-2,3,6)$, $\mathbf{a_4} = (-1,5,10)$, $\mathbf{a_5} = (4,1,2)$, and let $H = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \mathbf{a_4}, \mathbf{a_5}\}$. What is the rank of $H$? Add a non-zero vector to $H$ without changing the rank.

8. Let $V_1 = \{(t,t,t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$ and $V_2 = \{\lambda_1(1,0,2) + \lambda_2(-1,3,0) : \lambda_1, \lambda_2 \in \mathbb{R}\}$. Show that $V_1 \oplus V_2 = \mathbb{R}^3$. Split the vector $\mathbf{v} = (1,10,2)$ to components in $V_1$ and $V_2$.

# 4 Matrices

An $m \times n$ matrix $A$ is a rectangular array of $mn$ numbers arranged in $m$ rows and $n$ columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

We call the row vector $(a_{i1}, a_{i2}, \dots, a_{in})$ row $i$ and the column vector $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ column $j$. The $ij$-element or $ij$th component of $A$ is $a_{ij}$. In short, we write $A = (a_{ij})$, and we might specify the range of indices: $1 \leq i \leq m$ and $1 \leq j \leq n$. When $m = n$, the matrix is a *square matrix*. The entries of form $a_{ii}$ form the *main diagonal* of the matrix $A = (a_{ij})$.

Given a matrix $A$, we might interchange rows and columns to get the *transpose* of $A$. In notation, $A^T = (a_{ji})$ if $A = (a_{ij})$.

A square matrix is *upper triangular* if all its entries below the main diagonal are zero. It is *lower triangular* if all entries above the main diagonal are zero. A matrix is *diagonal* if all its non-zero entries lie in the main diagonal. In other words: $A = (a_{ij})$ is upper triangular if $a_{ij} = 0$ for $i > j$, lower triangular if $a_{ij} = 0$ for $i < j$, and diagonal if $a_{ij} = 0$ for $i \neq j$.

# Excercises

1. Calculate the entries of a $5 \times 5$ matrix such that the $ij$-element is $i^2 - 3j$.
2. Calculate the entries of a $6 \times 6$ matrix such that the $ij$-element is $-3i - 4j$.
3. Calculate the entries of a $6 \times 6$ matrix such that the $ij$-element is $i + j \pmod 6$.
4. Determine the $3 \times 4$ matrix $A$ such that $a_{ij} = 3i - j$. Give the transpose of $A$.

## 4.1 Matrix operations

**Addition of Matrices.** Let $A$ and $B$ be two matrices of the same size $m \times n$. We define the $m \times n$ matrix $A + B$ by adding the corresponding elements of $A$ and $B$.

**Example 4.1.** *The sum of two matrices.*

$$\begin{pmatrix} 2 & 3 & -1 & 4 \\ 5 & -3 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 3 & -3 \\ -1 & 2 & 3 & -1 \\ 2 & 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 & 1 \\ 4 & -1 & 3 & 1 \\ 3 & 2 & 0 & 5 \end{pmatrix}$$

**Multiplication by a scalar.** Let $A = (a_{ij})$ be an $m \times n$ matrix and $\beta$ a real number (scalar). Now the $m \times n$ matrix $\beta A$ is given such that the $ij$-element of $\beta A$ is $\beta a_{ij}$.

**Example 4.2.** *Let* $A = \begin{pmatrix} 4 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix}$. *Let us calculate* $3A - 2B$.

**Solution:** $3A - 2B = 3\begin{pmatrix} 4 & 0 & 2 \\ -1 & 3 & 1 \end{pmatrix} - 2\begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 6 \\ -3 & 9 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 4 & 2 \\ -2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -4 & 4 \\ -1 & 9 & -1 \end{pmatrix}$ $\square$

## Excercises

1. Let $A = \begin{pmatrix} -3 & 6 & -8 \\ -6 & 4 & 1 \\ -9 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -7 & 5 \\ -3 & -9 & -5 \\ 4 & 2 & 4 \end{pmatrix}$. Calculate $-9A + 6B$.

2. Let $A = \begin{pmatrix} 1 & 2 & 4 \\ -7 & 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 5 \\ 1 & -3 & 6 \end{pmatrix}$. Calculate $-2A + 3B$.

## 4.2 Matrix multiplication

A matrix can be thought of as a collection of row vectors and similarly column vectors. Therefore, it comes as no surprise that matrix multiplication is an iterated scalar product of vectors.
**Product of two matrices.** Let $A = (a_{ij})$ be an $m \times n$ matrix, and let $B = (b_{ij})$ be an $n \times p$ matrix. The product of $A$ and $B$ is an $m \times p$ matrix $C = (c_{ij})$, where

$$c_{ij} = (i\text{th row of } A) \cdot (j\text{th column of } B).$$

We can expand this to $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

Notice that two matrices can be multiplied together only if the number of columns in the first matrix equals the number of rows in the second matrix.

**Example 4.3.** *If* $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}$, *calculate* $AB$ *and* $BA$.

**Solution:** Since $A$ is a $2 \times 2$ matrix and so is $B$, their product $C = AB$ is a $2 \times 2$ matrix. If $C = (C_{ij})$, then we calculate $c_{11}$ as the dot product of the first row of $A$ and the first column of $B$.

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix} \quad \text{Thus } c_{11} = 3 + 15 = 18.$$

Similarly, to compute $c_{12}$ we do

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix} \quad \text{Thus } c_{12} = -2 + 18 = 16.$$

Continuing, we find $c_{21} = -6 + 20 = 14$ and $c_{22} = 4 + 24 = 28$. Therefore, $AB = \begin{pmatrix} 18 & 16 \\ 14 & 28 \end{pmatrix}$.

Similarly, we calculate $BA$ to get $\begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -7 & 39 \end{pmatrix}$.

This shows the important fact that matrix products do not commute in general. □

**Example 4.4.** *Let* $A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 1 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix}$. *Determine the rank of* $A$.

**Solution:** The rank of a matrix equals the rank of the vector set formed by the columns of the matrix. We use the basis transformation process to find the rank.

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{u_1}$ | 1 | 1 | 2 | 0 |
| $\mathbf{u_2}$ | 3 | 1 | 2 | 1 |
| $\mathbf{u_3}$ | 0 | 2 | 1 | 2 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | 1 | 2 | 0 |
| $\mathbf{u_2}$ | 0 | −2 | −4 | 1 |
| $\mathbf{u_3}$ | 0 | 2 | 1 | 2 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{a_3}$ | $\mathbf{a_4}$ |
|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | 1 | 2 | 0 |
| $\mathbf{a_4}$ | 0 | −2 | −4 | 1 |
| $\mathbf{u_3}$ | 0 | 6 | 9 | 0 |

Now we can include $\mathbf{a_2}$ in the basis and the rank is 3. □

In the class of square matrices, the following matrix plays an important role. Let $I_n$ denote the $n \times n$ *identity matrix* consisting of 1s in the main diagonal and 0 everywhere else. It has the property that $I_nA = AI_n = A$ for every $n \times n$ matrix $A$. In the class of $n \times n$ matrices, $I_n$ plays a role similar to that of 1 plays in the class of rational numbers. Therefore, it is important to ask the following question.

Given an $n \times n$ matrix $A$, is there another $n \times n$ matrix $B$ such that $AB = BA = I_n$? If the answer is yes, $B$ is the *inverse* of $A$ denoted as $A^{-1}$. However, this is not always the case. Some matrices do not have an inverse. There are several methods to calculate the inverse. We use one here, based on the basis exchange process. The key formula is the following. In the starting table, we use the columns of $A$, and we extend this by the vectors of the standard basis. This second part looks exactly as $I_n$. Now we try to include all $n$ column vectors in the bases by the basis exchange process. If we succeed, then the left part of the table is $I_n$. Or it can be made so by interchanging the rows. In that case, the extended part of the table shows us the inverse of $A$. In this case the rank of $A$ was $n$. However, if the rank of $A$ is smaller, then we cannot include all vectors in the basis. Therefore, $A$ does not have an inverse. It also means that the determinant of $A$ is 0. Let us see a small example.

**Example 4.5.** *Let* $A = \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$. *Compute the inverse of $A$ if it exists.*

**Solution:** We use the basis transformation process.

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{u_1}$ | $\mathbf{u_2}$ |
|---|---|---|---|---|
| $\mathbf{u_1}$ | 2 | $-3$ | 1 | 0 |
| $\mathbf{u_2}$ | $-4$ | 5 | 0 | 1 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{u_1}$ | $\mathbf{u_2}$ |
|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | $-3/2$ | $1/2$ | 0 |
| $\mathbf{u_2}$ | 0 | $-1$ | 2 | 1 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{u_1}$ | $\mathbf{u_2}$ |
|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | 0 | $-5/2$ | $-3/2$ |
| $\mathbf{a_2}$ | 0 | 1 | $-2$ | $-1$ |

We can check the solution:

$$\begin{array}{cc} & \begin{pmatrix} -5/2 & -3/2 \\ -2 & -1 \end{pmatrix} \\ \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} . \square$$

**Example 4.6.** *Let* $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$. *Compute the inverse of $A$ if it exists.*

**Solution:** We use the basis transformation process.

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{u_1}$ | $\mathbf{u_2}$ |
|---|---|---|---|---|
| $\mathbf{u_1}$ | 1 | 2 | 1 | 0 |
| $\mathbf{u_2}$ | $-2$ | $-4$ | 0 | 1 |

$\longrightarrow$

| basis | $\mathbf{a_1}$ | $\mathbf{a_2}$ | $\mathbf{u_1}$ | $\mathbf{u_2}$ |
|---|---|---|---|---|
| $\mathbf{a_1}$ | 1 | 2 | 1 | 0 |
| $\mathbf{u_2}$ | 0 | 0 | 2 | 1 |

This is as far as we can go. We deduce that the rank of $A$ is 1, therefore $A$ is not invertible. $\square$

# Exercises

1. Compute the following

$$\begin{pmatrix} 1 & 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & -6 \\ 2 & 4 \\ 1 & 0 \\ -2 & 3 \end{pmatrix}.$$

2. Let $A = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$.

   Compute $AB$, $AC$ and $CA$. Explain what you get.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 \end{pmatrix}$ and $C = \begin{pmatrix} -6 & -5 & -4 & -3 \\ -2 & -1 & 0 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

   Calculate the following products, if they exist: $AB$, $BA$, $CB^T$, $BC$.

4. Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & -3 \\ 1 & -1 & 2 \\ 3 & -2 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 2 \\ 4 & -1 \end{pmatrix}$

   Calculate the following products, if they exist: $AC$, $CA$, $A^T B^T$, $AB$, $C^T B$, $BA^T$.

5. $A := \begin{pmatrix} 2 & -5 & 4 \end{pmatrix}$ $B := \begin{pmatrix} 3 & 1 & 0 \\ -2 & 2 & 5 \\ 4 & 1 & -3 \end{pmatrix}$ $C := \begin{pmatrix} 2 \\ -4 \\ 7 \end{pmatrix}$

   Calculate the following products: $A(BC)$ and $(AB)C$.

6. Let $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 1 & 5 \end{pmatrix}$, és $C = \begin{pmatrix} 0 & -2 & 1 \\ 4 & 3 & 2 \\ -5 & 0 & 6 \end{pmatrix}$

   Calculate the following products: $AB$, $BC$, $A(BC)$ and $(AB)C$.

7. Let $A = \begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 0 & -1 & 1 \end{pmatrix}$. Calculate $A^{-1}$ if it exists.

8. Let $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}$. Calculate $A^{-1}$ if it exists.

9. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$. If $A$ is invertible, find the inverse!

10. $B := \begin{pmatrix} 3 & 1 & 0 \\ -2 & 2 & 5 \\ 4 & 1 & -3 \end{pmatrix}$

    Is $B$ invertible? If yes, determine the inverse matrix!

11. $A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $B := \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

    Is $A^T$ or $B$ invertible? If yes, determine the inverse matrix!

# 5 Determinants

Determinants were first used to determine the solution of a system of linear equations. However, we first learnt about matrices and now it is easier to imagine that the determinant is a number associated to a matrix. First of all, if the matrix is diagonal, then its determinant is the product of the elements in the main diagonal.

**Example 5.1.** *Let* $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$, *then* $det(A) = -3$. □

Also, if the matrix is upper (or lower) triangular, then its determinant is the product of the elements in the main diagonal.

**Example 5.2.** *Let* $B = \begin{pmatrix} 3 & 0 & 0 \\ 12 & -2 & 0 \\ \pi & 1/e & 1/2 \end{pmatrix}$, *then* $det(B) = -3$. □

To distinguish determinants from matrices, we use vertical lines around the array of numbers if we denote a determinant opposed to brackets in case of matrices. However, the same elementary row operations that we used for matrices (in Section 1), can be used for calculating the determinant. The following rules apply.

(i) The determinant is unchanged if we add a multiple of a row (column) to another row (column).
(ii) We can factor out a common divisor from any row or column.
(iii) If we interchange two rows (columns), the sign of the determinant changes.

In what follows, we can use the following strategy to calculate a determinant: we use elementary row operations to transform our original matrix to a triangular matrix. At the end, we easily calculate the final determinant.

**Example 5.3.** *Calculate the determinant* $|A| = \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 2 & 1 & 9 & 6 \\ 3 & 2 & 4 & 8 \end{vmatrix}$.

**Solution:** There is already a 0 in the first column, so it is simplest to reduce the other elements of the first column to 0. We continue aiming for an upper triangular matrix.

Multiply the first row by $-2$ and add it to the third row and multiply the first row by $-3$ and add it to the fourth row.

$$|A| = \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 2 & 1 & 9 & 6 \\ 3 & 2 & 4 & 8 \end{vmatrix}$$

Multiply the second row by $-5$ and $-7$ and add it to the third and fourth rows, respectively.

$$= \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -16 & -18 \\ 0 & 0 & -32 & -26 \end{vmatrix}$$

Subtract the third row twice from the fourth

$$= \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -16 & -18 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

Now we have an upper triangular matrix and $|A| = (-1)(-16)10 = 160$. □

**Example 5.4.** *Calculate the determinant* $|A| = \begin{vmatrix} 1 & -2 & 3 & -5 & 7 \\ 2 & 0 & -1 & -5 & 6 \\ 4 & 7 & 3 & -9 & 4 \\ 3 & 1 & -2 & -2 & 3 \\ -5 & -1 & 3 & 7 & -9 \end{vmatrix}$.

**Solution:** Adding row 2 and then row 4 to row 5, we obtain $|A| = \begin{vmatrix} 1 & -2 & 3 & -5 & 7 \\ 2 & 0 & -1 & -5 & 6 \\ 4 & 7 & 3 & -9 & 4 \\ 3 & 1 & -2 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0.$

This illustrates that a little looking before doing all the computations can simplify the matters considerably. □

**Example 5.5.** *Let* $C = \begin{vmatrix} -3 & -3 & 6 \\ -5 & 7 & 0 \\ -5 & -3 & -3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 & -2 \\ -5 & 7 & 0 \\ -5 & -3 & -3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 & -2 \\ -5 & 7 & 0 \\ 0 & -10 & -3 \end{vmatrix} =$

$-3 \begin{vmatrix} 1 & 1 & -2 \\ 0 & 12 & -10 \\ 0 & -10 & -3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 & -2 \\ 0 & 2 & -13 \\ 0 & -10 & -3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 & -2 \\ 0 & 2 & -13 \\ 0 & 0 & -68 \end{vmatrix} = (-3)2(-68) = 408.$ □

**Example 5.6.** $\begin{vmatrix} 2 & -3 & 5 \\ 1 & 7 & 2 \\ -4 & 6 & -10 \end{vmatrix} = 0$, *since the third row is* $-2$ *times the first row.*

So far we calculated the determinant without actually defining it. We fill this gap now. It should be apparent why this delay was useful. A **rook placement** in a matrix of order $n$ is a set of $n$ entries, one from each row and column. Given a matrix $A$ of order $n$, we define the **determinant** as a signed sum of products of entries in all rook placements. The number of rook placements is $n!$, which grows exponentially with $n$. The sign of a product (in the evaluation of the determinant) is calculated as follows. Given the entries of a rook placement, we associate a permutation. For instance the rook placement $\{a_{11}, a_{22}, \ldots, a_{88}\}$ corresponds to the permutation $123 \ldots 8$. This monotone increasing permutation has always positive sign. The rook placement $\{a_{13}, a_{24}, a_{31}, a_{47}, a_{55}, a_{62}, a_{78}, a_{86}\}$ corresponds to the permutation 34175286, that is, we only list the second indeces. Now we have to determine the number of inversions that transforms 34175286 to 12345678. If this number is $k$, then the sign of the product is $(-1)^k$ in the evaluation of the determinant. In our case the number of inversions is 9. Therefore, the product $a_{13}a_{24}a_{31}a_{47}a_{55}a_{62}a_{78}a_{86}$ will have a negative sign in the summation.

**Example 5.7.** *Let us calculate the following* $2 \times 2$ *determinant:* $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

**Solution:** The product of the entries in the main diagonal has always positive sign. The other diagonal corresponds to the permutation (21). Therefore we need 1 inversion, hence the product has negative sign. $D = ad - bc$. □

**Example 5.8.** *Let us calculate the following* $3 \times 3$ *determinant:* $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

**Solution:** There are 6 rook placements, so the following products must be present in the evaluation of the determinant: $a_{11}a_{22}a_{33}$, $a_{11}a_{23}a_{32}$, $a_{12}a_{21}a_{33}$, $a_{12}a_{23}a_{31}$, $a_{13}a_{22}a_{31}$, $a_{13}a_{21}a_{32}$. Now the signs are respectively $1, -1, -1, 1, -1, 1$. Therefore the determinant is: $a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$. □

There is one more method to calculate the determinant. The philosophy behind this method is the fact that smaller determinants are easier to calculate.

Let $B = (b_{ij})$ be an $n \times n$ matrix. The $(n-1) \times (n-1)$ matrix arising from $B$ by deleting row 2 and column 5 is denoted by $B_{25}$.

The expansion of $det(B)$ in row $i$ is the following summation:

$$det(B) = a_{i1}(-1)^{i+1}|B_{i1}| + a_{i2}(-1)^{i+2}|B_{i2}| + \cdots + a_{in}(-1)^{i+n}|B_{in}|.$$

This is also known as Laplace expansion or expansion by cofactors in row $i$. Similarly, we can expand a determinant in a column.

**Example 5.9.** *Let* $A = \begin{pmatrix} 3 & 5 & 2 \\ 4 & 2 & 3 \\ -1 & 2 & 4 \end{pmatrix}$. *Let us calculate the determinant by expanding in the second row or the third column.*

**Solution:** $det(A) = 4(-1)^3|A_{21}| + 2(-1)^4|A_{22}| + 3(-1)^5|A_{23}|$. Now $|A_{21}| = \begin{vmatrix} 5 & 2 \\ 2 & 4 \end{vmatrix} = 5*4 - 2*2 = 16$, $|A_{22}| = \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} = 3*4 - (-1)*2 = 14$, and $|A_{23}| = \begin{vmatrix} 3 & 5 \\ -1 & 2 \end{vmatrix} = 3*2 - (-1)*5 = 11$.

Therefore $det(A) = -4*16 + 2*14 - 3*11 = -69$.

Similarly, if we expand in the third column, say, we obtain:

$|A_{13}| = \begin{vmatrix} 4 & 2 \\ -1 & 2 \end{vmatrix} = 4*2 - (-1)*2 = 10$, $|A_{33}| = \begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = 3*2 - 4*5 = -14$.

Therefore, $det(A) = 2*10 - 3*11 + 4*(-14) = -69$. □

The following is known as the multiplicativity of the determinant.

**Fact 5.10.** $det(AB) = det(A)det(B)$.

**Example 5.11.** *Verify the previous fact for* $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 0 & -2 & 5 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ 2 & 0 & -2 \end{pmatrix}$.

**Solution:** We first calculate $det(A) = 5 - 12 + 8 + 15 = 16$ and $det(B) = 2 - 16 + 6 = -8$. After that we calculate $AB = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -1 & -5 \\ 11 & -7 & 5 \\ 10 & 2 & -18 \end{pmatrix}$. Now $det(AB) = 630 - 110 - 50 - 350 - 50 - 198 = -128$, which is really $16(-8) = det(A)det(B)$. □

# Exercises

1. What is the sign of the following products in the evaluation of a determinant of order 6?

   $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$, $\quad a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$

2. Using only the definition of the determinant, show that the determinant below is 0.

   $$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ e & f & 0 & 0 & 0 \end{vmatrix}$$

3. Calculate the coefficients of $x^3$ and $x^4$ in the expression below, using the definition of the determinant. $f(x) = \begin{vmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$

4. The $3 \times 3$ Vandermonde determinant is given by

   $D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix}$. Show that $D_3 = (a_2 - a_1)(a_3 - a_2)(a_3 - a_1)$.

5. The $4 \times 4$ Vandermonde determinant is given by

   $D_4 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{vmatrix}$. Show that $D_4 = (a_2 - a_1)(a_3 - a_2)(a_3 - a_1)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3)$.

6. In each example below, evaluate the determinant using the methods of this chapter.

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 0 & 6 \\ 5 & -2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & -1 & 2 & 1 \\ 4 & 3 & 1 & -2 \\ -1 & 0 & 2 & 3 \\ 6 & 2 & 5 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix},$$

$$\begin{vmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & -6 & 4 & 0 \end{vmatrix}, \begin{vmatrix} 6 & 1 & -9 & 1 \\ -8 & -6 & 3 & 0 \\ -6 & -8 & -5 & 0 \\ 2 & 7 & 3 & 0 \end{vmatrix}.$$

7. Calculate the following determinant using row expansions:

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & a & 0 & 0 \\ 1 & 1 & b & 0 \\ 1 & 0 & 1 & c \end{vmatrix}$$

8. Calculate the following determinant using elementary row/column operations:

$$\begin{vmatrix} 2 & 8 & 6 & 4 \\ 0 & 1 & 3 & 0 \\ 6 & 1 & 6 & 9 \\ 9 & 9 & 1 & 9 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 5 & -1 \\ -4 & 1 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{vmatrix}, \begin{vmatrix} 2 & -1 & 0 & 2 \\ -4 & 2 & -9 & 3 \\ 2 & -6 & 4 & -2 \\ 1 & 3 & 2 & 2 \end{vmatrix}$$

9. Calculate the following determinant by expanding in the first column:

$$\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}$$

## 5.1 Cramer's rule

We can write a system of $n$ equations in $n$ unknowns in the following concise form:

$A\mathbf{x} = \mathbf{b}$.

Here $A$ denotes an $n \times n$ matrix, $\mathbf{x}$ and $\mathbf{b}$ are $n$-dimensional column vectors.

If $det(A) \neq 0$, then the system has a unique solution given by Cramer's rule:

$$x_1 = \frac{D_1}{det(A)}, \quad x_2 = \frac{D_2}{det(A)}, \quad \dots \quad x_n = \frac{D_n}{det(A)},$$

where $D_j$ is the determinant of the matrix obtained by replacing the $j$th column of $A$ by the vector $\mathbf{b}$.

**Example 5.12.** *Let us solve, using Cramer's rule, the following system:*

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 6x_3 & = & 18 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 24 \\ 3x_1 & + & x_2 & - & 2x_3 & = & 4 \end{array}$$

**Solution:** First we calculate the determinant of the matrix formed by the coefficients of the equations. $D = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 6 \neq 0$, so the system has a unique solution. Now replacing the first column, we get $D_1 = \begin{vmatrix} 18 & 4 & 6 \\ 24 & 5 & 6 \\ 4 & 1 & -2 \end{vmatrix} = 24$, and similarly $D_2 = \begin{vmatrix} 2 & 18 & 6 \\ 4 & 24 & 6 \\ 3 & 4 & -2 \end{vmatrix} = -12$, and $D_3 = \begin{vmatrix} 2 & 4 & 18 \\ 4 & 5 & 24 \\ 3 & 1 & 4 \end{vmatrix} = 18$. Therefore, $x_1 = \frac{D_1}{D} = \frac{24}{6} = 4$, $x_2 = -\frac{12}{6} = -2$, and $x_3 = \frac{D_3}{D} = \frac{18}{6} = 3$. We can check the solution by substituting it to the equations. For instance, into the third: $3 \cdot 4 - 2 - 2 \cdot 3 = 4$. □

Of course, we remember from Section 1, that some systems of linear equations might have 0 or infinitely many solutions. In Cramer's rule, these two events must be covered by the case, where $det(A) = 0$. If we find that some $D_i \neq 0$, then the system has no solutions. Finally, if $det(A) = D_1 = \cdots = D_n = 0$, then there are infinitely many solutions. However, Cramer's rule does not give a recipe how to find them.

**Example 5.13.** *Let us solve, using Cramer's rule, the following system:*

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 6 \\ 3x_1 & - & 2x_2 & + & 5x_3 & = & 3 \\ 6x_1 & + & x_2 & + & 2x_3 & = & 21 \end{array}$$

**Solution:** First, we calculate the determinant of the matrix formed by the coefficients of the equations. $D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 5 \\ 6 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -5 & 8 \\ 6 & -5 & 8 \end{vmatrix} = 40 \begin{vmatrix} 1 & 0 & 0 \\ 3 & -1 & 1 \\ 6 & -1 & 1 \end{vmatrix} = 0$. We first subtracted column 1 from column 2 and added column 1 to column 3. Secondly we factored out 5 from column 2 and 8 from column 3.

Now let us calculate the modified determinants:

$D_1 = \begin{vmatrix} 6 & 1 & -1 \\ 3 & -2 & 5 \\ 21 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 6 & 1 & -1 \\ 3 & -2 & 5 \\ 3 & -2 & 5 \end{vmatrix} = 0$, where we subtract 3 times row 1 from row 3.

$D_2 = \begin{vmatrix} 1 & 6 & -1 \\ 3 & 3 & 5 \\ 6 & 21 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 6 & -1 \\ 3 & 3 & 5 \\ 3 & 3 & 5 \end{vmatrix} = 0$, where we subtract 3 times row 1 from row 3.

$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & 3 \\ 6 & 1 & 21 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & 3 \\ 3 & -2 & 3 \end{vmatrix} = 0$. We conclude that the system has an infinite number of solutions. □

**Example 5.14.** *Let us solve, using Cramer's rule, the following system:*

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 4 \\ 2x_1 & - & 3x_2 & + & x_3 & = & -5 \\ 4x_1 & - & x_2 & - & x_3 & = & -3 \end{array}$$

**Solution:** First, we calculate the determinant of the matrix formed by the coefficients of the equations. $D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 4 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{vmatrix} = 0$. We subtracted multiples of row 1 from row 2 and 3.

Now we have to calculate the modified determinants:

$D_1 = \begin{vmatrix} 4 & 1 & -1 \\ -5 & -3 & 1 \\ -3 & -1 & -1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 4 \\ 1 & -3 & -5 \\ -1 & -1 & -3 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 4 \\ 0 & -2 & -1 \\ 0 & -4 & -8 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{vmatrix} = 12$. This shows that the system has no solutions. □

# Exercises

Solve the following system of equations using Cramer's rule.

1. $\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & = & 6 \\ 3x_1 & - & 2x_2 & - & 3x_3 & = & 5 \\ 8x_1 & + & 2x_2 & + & 5x_3 & = & 11 \end{array}$

2. $\begin{array}{rcrcrcr} 2x_1 & + & 5x_2 & - & x_3 & = & -1 \\ 4x_1 & + & x_2 & + & 3x_3 & = & 3 \\ -2x_1 & + & 2x_2 & & & = & 0 \end{array}$

3. $\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & - & 2x_4 & = & 6 \\ 2x_1 & - & x_2 & - & 2x_3 & - & 3x_4 & = & 8 \\ 3x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & = & 4 \\ 2x_1 & - & 3x_2 & + & 2x_3 & + & x_4 & = & -8 \end{array}$

$$4.\quad \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 5 \\ 2x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & = & 1 \\ 3x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & = & 1 \\ 4x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & = & -5 \end{array}$$

## 5.2 Vector product

As it was promised in Section 2.1, we now define a useful notion. Suppose there are two given independent vectors **a** and **b** in 3 dimensions. We would like to find a vector, that is orthogonal to both **a** and **b**. The solution is the cross product or vector product $\mathbf{a} \times \mathbf{b}$. It has various applications in Physics. We use the so called right-hand rule. If we open three fingers on our right hand, and **a** lies along our index finger, **b** lies along our middle finger, then $\mathbf{a} \times \mathbf{b}$ points along our thumb. Using coordinates $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, the vector product is formally given by the following determinant:

$$\begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

It yields $\mathbf{e_1}(a_2b_3 - b_2a_3) - \mathbf{e_2}(a_1b_3 - b_1a_3) + \mathbf{e_3}(a_1b_2 - b_1a_2) = (a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

# 6 Lines and planes in space

We remember from high school that a linear equation in two variables corresponds to a line in the Cartesian plane. That is, the solution set is a line. If we have two equations, then they correspond to two lines. Geometrically, we know that two lines in the plane either intersect in one point or parallel (may also coincide as a special case). Making a step up, we can consider three variables. The solutions of a linear equation corresponds to a plane in the three-dimensional Cartesian system. If there is a system of equations, then we have to study the common solutions of the equations. Geometrically, we know that two planes either intersect or parallel in 3D. However, now we might have more than two equations and some solution sets might be degenerate and correspond to lines. Therefore, we study the geometric objects, lines and planes in space and their corresponding equations.

A line is determined by a point $P = (p, r, q)$ and its direction vector $\underline{v} = (v_x, v_y, v_z)$. The points of the line are given by the following coordinates: $x = p + tv_x$, $y = r + tv_y$, $z = q + tv_z$, where $t \in \mathbb{R}$. Any triple $(x, y, z)$ is a point of the given line, and these are all. These are the *parametric equations* of the line. If we express the parameter $t$ from each equation, it yields the symmetric equations of the line $t = \frac{x-p}{x_v} = \frac{y-r}{v_y} = \frac{z-q}{v_z}$. However, this only makes sense if the coordinates of $\underline{v}$ are non-zero.

**Example 6.1.** *Determine the parametric equations and the symmetric equations of the line passing through the point $P_0$ and parallel to the vector $\underline{v}$.*

(a) $P_0 = (6, 1, -3)$ *and* $\underline{v} = (2, -3, 5)$.

(b) $P_0 = (6, 1, 0)$ *and* $\underline{v} = (0, -3, 5)$.

(c) $P_0 = (1, 1, 4)$ *and* $\underline{v} = (0, -3, 0)$.

**Solution:** (a) The parametric equations are as follows: $x = 6 + 2t$, $y = 1 - 3t$, $z = -3 + 5t$.
Solving the above equations for $t$, we get the symmetric equations: $t = \frac{x-6}{2} = \frac{y-1}{-3} = \frac{z+3}{5}$.

(b) The parametric equations are as follows: $x = 6$, $y = 1 - 3t$, $z = 5t$.
Since the first direction coordinate is 0, we get: $x = 6$ and $\frac{y-1}{-3} = \frac{z+3}{5}$.

(c) The parametric equations are the following: $x = 1$, $y = 1 - 3t$, $z = 4$.
There is no symmetric equation, and the line $x = 1$, $z = 4$ is parallel to the $y$-axis. □

We learnt in Elementary Geometry that two points determine a line. Let us see, how this applies to our tools in Linear Algebra.

**Example 6.2.** *Let $A = (2, 2, 3)$ and $B = (1, 3, 3)$ be two points in space. Determine the parametric equations of the line through $A$ and $B$.*

**Proof:** To determine the parametric equations, we need the direction of the line. Now $\underline{v} = \overrightarrow{AB} = (-1, 1, 0)$. Using the point $A$ and the vector $\underline{v}$, we get the parametric equations: $x = 2 - t$, $y = 2 + t$, $z = 3$. □

Next, we consider planes. There is a fundamental object that helps us to describe the position of a plane. This is the *normal vector*, usually denoted by $\underline{n}$. It has the property that every vector of the plane is perpendicular (orthogonal) to $\underline{n}$. Given the normal vector, there are infinitely many parallel planes having this property. Giving the coordinates of a point lying in the plane distinguishes the plane uniquely. Therefore, given a point of the plane $P = (p, q, r)$ and the normal vector $\underline{n} = (n_x, n_y, n_z)$, the points of the plane are the solutions $(x, y, z)$ of the following equation: $(x - p)n_x + (y - q)n_y + (z - r)n_z = 0$ using the scalar product. Sometimes we write this in a different form, moving the constants to the right-hand side: $xn_x + yn_y + zn_z = pn_x + qn_y + rn_z$.

Let us see a few typical scenarios.

**Example 6.3.** *Find the plane $\pi$ passing through the point $P = (5, 2, 3)$ having normal vector $\underline{n} = (1, 3, -1)$. Do $A = (4, 1, -1)$ or $B = (2, 2, 2)$ belongs to $\pi$?*

**Solution:** We obtain the equation of the plane as follows: $(x-5)+3(y-2)-(z-3) = 0$ or equivalently $x + 3y - z = 8$. Substituting the coordinates of $A$ and $B$ shows that $A$ belongs to $\pi$ and $B$ does not. □

**Example 6.4.** *The equation of a plane is $3x - 2y + 4z = 14$. Find a normal vector and a couple of points of the plane.*

**Solution:** We see that $\underline{n} = (3, -2, 4)$ is a normal vector. We find points of the plane by setting two coordinates and calculating the third one from the equation. For instance, let $x = 2$ and $y = 2$, then $6 - 4 + 4z = 14$ yields $z = 3$. That is, $(2, 2, 3)$ belongs to the plane. Similarly setting $y = 1$ and $z = 3$ yields $3x - 2 + 12 = 14$. That is, $(3/4, 1, 3)$ belongs to the plane as well. □

**Example 6.5.** *Find the equation of a plane $\pi$ that is orthogonal to the line $e$ given by $\frac{x-4}{3} = \frac{y}{-2} = z$ and containing the point $P(4, 0, -1)$.*

**Solution:** Since the normal vector of the plane $\pi$ is parallel to the line $e$, we deduce $\underline{n} = (3, -2, 1)$. Now the equation of $\pi$ is $3(x - 4) - 2y + (z + 1) = 0$, equivalently $3x - 2y + z = 11$. □

**Example 6.6.** *Find the equation of the plane $\pi$ containing the line $e : \frac{x-2}{2} = \frac{y-1}{-1}$, $z = 2$ and containing the point $P = (4, 5, 3)$.*

**Solution:** We check by substitution that $P$ does not belong to $e$. Therefore, there is precisely one solution. We need to determine the normal vector of the plane. For this, we find two vectors spanning the plane. One of them is the direction of $e$, that is, $\underline{v} = (2, -1, 0)$. For a second one, we use $P$ and a point $P_0$ of the line $e$. We see that $P_0 = (2, 1, 2)$ is a point of $e$. Now $\overrightarrow{P_0P} = (2, 4, 1)$. The normal vector we seek must be orthogonal to both $\underline{v}$ and $\overrightarrow{P_0P}$. Such a vector is given by the vector product of $\underline{v}$ and $\overrightarrow{P_0P}$.

$$\underline{n} = \underline{v} \times \overrightarrow{P_0P} = (2, -1, 0) \times (2, 4, 1) = (-1, -2, 10).$$

Therefore, the equation of $\pi$ is: $-(x - 4) - 2(y - 5) + 10(z - 3) = 0$, equivalently $-x - 2y + 10z = 16$. □

In Euclidean Geometry we learnt that three non-collinear points determine a plane. We must be able to solve such a problem with our machinery.

**Example 6.7.** *Let $P_1 = (8, 6, -2)$, $P_2 = (4, -3, -7)$, $P_3 = (-2, 2, 3)$, be three points in the space. What is the equation of the plane through $P_1, P_2, P_3$?*

**Solution:** Let the plane that we seek be called $\pi$. We need a point and a normal vector to determine the equation of $\pi$. We have a point, $P_1$ say. What is a normal vector? We know that a normal vector is orthogonal to the vectors of $\pi$. We can determine two independent vectors in $\pi$ using the three points. $\overrightarrow{P_1P_2}$ is a good vector and so is $\overrightarrow{P_1P_3}$. Now $\overrightarrow{P_1P_2} = (-4, -9, -5)$ and $\overrightarrow{P_1P_3} = (-10, -4, 5)$. We now recall that the vector product of two vectors is orthogonal to the two vectors. Therefore, $\underline{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-4, -9, -5) \times (-10, -4, 5) = ((-9) \cdot 5 - (-4) \cdot (-5), -(-10) \cdot (-5) - (-4) \cdot 5, (-4) \cdot (-4) - (-10) \cdot (-9)) = (-65, 70, -74)$.

Now the equation of $\pi$ can be written as $-65(x - 8) + 70(y - 6) - 74(z + 2) = 0$. Equivalently $-65x + 70y - 74z = 48$ □

## Excercises

1. Let $P_0 = (3, 1, -4)$ and $\underline{v} = (4, 5, 0)$. Find the parametric and symmetric equations of the line through $P_0$ and parallel to $\underline{v}$.

2. Let $P_1 = (1, 4, 5)$ and $P_2 = (3, 6, -1)$. Find the parametric and symmetric equations of the line through $P_1$ and $P_2$.

3. Find the direction vector and a point of the following lines. Convert to symmetric equations.

   e: $x = 2 + 3t$, $y = -1 + 2t$, $z = 5 - 4t$.

   f: $x = 5t$, $y = -2 + 7t$, $z = 4$.

   g: $x = 6$, $y = 1 + 3t$, $z = 0$.

4. Convert the following symmetric equations to parametric equations:
$\frac{x-3}{-5} = \frac{z+1}{7}$, and $y = 4$.

5. Convert the following symmetric equations to parametric equations:
$\frac{x-1}{2} = \frac{y+1}{3} = z - 4$,

6. The following plane is given. $\pi : 2x - 3y + 5z - 5 = 0$. Find the normal vector and a couple of points. Are the points $(-8, 3, 6)$ and $(1, 4, -3)$ incident to $\pi$?

7. Determine the plane through $(2, -1, 4)$ having normal vector $(2, 3, -1)$.

8. Find the plane orthogonal to the line $e : \frac{x-4}{2} = y = \frac{z+2}{3}$ and through $(5, -1, 0)$.

9. Find the plane through the following points: $(3, -2, 1)$, $(-8, -7, 0)$, $(-7, -2, 7)$.

10. Find the plane through the point $(8, 3, -5)$ and containing the line $\frac{x+3}{-4} = \frac{y+4}{-2} = \frac{z-7}{-1}$.

## 6.1 Intersection of lines and planes

We first study the case, when two lines are given in the space. Geometrically three things can happen: the two lines intersect, the two lines are parallel (or even coincide), or the two lines are skew. This last situation did not occur in the plane.

**Example 6.8.** *Let the following lines be given:*

$$e : \begin{cases} x = & 1+ & 2t \\ y = & 3- & t \\ z = & 2+ & t \end{cases}, \quad g : \begin{cases} x = & & -6t \\ y = & 5+ & 3t \\ z = & 1- & 3t \end{cases}, \quad h : \begin{cases} x = & 4+ & t \\ y = & 2- & t \\ z = & 1+ & 3t \end{cases}.$$

*Determine the relative position of any two lines. In case of intersecting lines, find the intersection point.*

**Solution:** The direction vector of $e$ is $\underline{v}_e = (2, -1, 1)$, which is parallel to the direction vector of $g$: $(-6, 3, -3)$. Therefore, these two lines are either parallel or identical. We check an arbitrary point of $e$, whether it lies on $g$ or not. Now $(1, 3, 2)$ lies on $e$, bot not on $g$. Therefore, these two lines are parallel.

The direction vector of $e$ is $\underline{v}_e = (2, -1, 1)$ and the direction vector of $h$ is $\underline{v}_h = (1, -1, 3)$. These vectors are not parallel. Therefore the two lines are either skew or intersecting in a point. We try to determine the intersection using the parametric equations of the lines. We distinguish the parameters $t_1$ and $t_2$ for the two lines and try to find identical triples $(x, y, z)$ given by the parametric equations.

$$\begin{array}{ccccccc} 1 & + & 2t_1 & = & 4 & + & t_2 \\ 3 & - & t_1 & = & 2 & - & t_2 \\ 2 & + & t_1 & = & 1 & + & 3t_2 \end{array}$$

The second equation yields $t_1 = 1 + t_2$. Substituting this in the third equation, we get $t_2 = 1$. Therefore $t_1 = 2$. We have to check, whether these values satisfy the so far unused first equation. We get $5 = 5$, correct. Hence, the intersection point is $(5, 1, 4)$.

The direction vector of $g$ is $\underline{v}_g = (-6, 3, -3)$, the direction vector of $h$ is $\underline{v}_h = (1, -1, 3)$. Since these vectors are not parallel, we have to check, whether or not they intersect. We have to use the parametric equations of the lines. We distinguish the parameters $t_1$ and $t_2$ for the two lines and try to find identical triples $(x, y, z)$ given by the parametric equations.

$$\begin{array}{ccccccc} & & -6t_1 & = & 4 & + & t_2 \\ 5 & - & 3t_1 & = & 2 & - & t_2 \\ 1 & - & 3t_1 & = & 1 & + & 3t_2 \end{array}$$

Adding twice the second equation to the first one, we eliminate $t_1$ and get $t_2 = -2$. This gives us $t_1 = -\frac{1}{3}$. However, substitution of these two values in the third equation results in a contradiction. Therefore, there is no intersection, the two lines are skew. □

In the second part, we consider a plane and a line in space. Geometrically three things can happen: either the line lies entirely in the plane or they have one common point, or zero. That is the line is parrallel to the plane.

**Example 6.9.** *Let the following plane and lines be given:* $\pi: \quad 2x-y+3z{=}16$, $e: \quad x-2=\frac{y}{2}$ *and* $z=4$, $f: \quad \frac{x-3}{-1}=y+5=z-4$. *What is the relative position of* $\pi$ *to* $e$ *and* $f$*? If they intersect, determine the intersection point.*

**Solution:** The direction vector of $e$ is $\underline{v}_e=(1,2,0)$, and a normal vector of $\pi$ is $\underline{n}=(2,-1,3)$. We check the scalar product of these two vectors: $\underline{v}_e\cdot\underline{n}=1*2-1*2+0=0$. That is, these two vectors are orthogonal. Now, either $e$ is parallel to $\pi$ or $e$ lies in $\pi$. Since $P=(2,0,4)$ is a point of $e$, and also satisfies the equation of $\pi$, the line $e$ lies in $\pi$.

The direction vector of $f$ is $\underline{v}_f=(-1,1,1)$, and a normal vector of $\pi$ is $\underline{n}=(2,-1,3)$. We check the scalar product of these two vectors: $\underline{v}_f\cdot\underline{n}=-1*2+1*(-1)+1*3=0$. That is, these two vectors are orthogonal. Now, either $f$ is parallel to $\pi$ or $f$ lies in $\pi$. Since $Q=(3,-5,4)$ is a point of $f$, but does not satisfy the equation of $\pi$, therefore $f$ and $\pi$ are parallel. □

Finally, we consider two planes in space. Geometrically two different planes either intersect or not. In the former case, their intersection is a line. In the latter case, the two planes are parallel. That is, the two normal vectors are parallel.

**Example 6.10.** *Let the following two planes be given:* $\pi_1: \quad 2x-y+4z=9$ *and* $\pi_2: \quad x+3y-z=2$. *Determine the parametric equations of the line of intersection of* $\pi_1$ *and* $\pi_2$.

**Solution:** One can check that the two normal vectors are not parallel. Therefore, the two planes really intersect. To find the parametric equations of the line of intersection, we need a point and a direction vector. The coordinates of any such point must satisfy both equations $2x-y+4z=9$ and $x+3y-z=2$. We have the freedom to choose one coordinate, since there are 3 equations and only two unknowns. Let us set $x=1$. After that we find $y=1$ and $z=2$. That is, $P_0=(1,1,2)$ is a point in the intersection.

The direction vector of the intersection line is orthogonal to both normal vectors. One such vector is the cross product: $\underline{n}_1\times\underline{n}_2=(2,-1,4)\times(1,3,-1)=(-11,6,7)$. Using this, we get the parametric equations:

$$e: \begin{array}{rrr} x= & 1- & 11t \\ y= & 1+ & 6t \\ z= & 2+ & 7t \end{array} \quad \square$$

# Excercises

1. Let the following lines be given:
$$e:\left\{\begin{array}{rrr} x= & -1+ & t \\ y= & & 2t \\ z= & 1- & 3t \end{array}\right., \quad f:\left\{\begin{array}{rrr} x= & & 3t \\ y= & 2+ & t \\ z= & -2+ & 5t \end{array}\right., \quad g:\left\{\begin{array}{rrr} x= & & -2t \\ y= & 5- & 4t \\ z= & 1+ & 6t \end{array}\right..$$
Determine the relative position of $e$ to $f$, similarly $e$ to $g$, and $f$ to $g$. In case of intersecting lines, find the intersection point.

2. Let $\pi: 2x-4y+6z=6$ be a given plane and $e: \frac{x-3}{2}=y=2z-3$ be a given line. What is the relative position of $\pi$ and $e$? Find the intersection points, if there are any.

3. Let $\pi: 8x-4y+7z=-7$ be a given plane and $a: \frac{x+7}{-1}=\frac{y+8}{-5}=\frac{z+9}{4}$ a given line. What is the relative position of $\pi$ and $a$? Find the intersection points, if there are any.

4. Let the following two planes be given:
$\pi_1: 2x-5y+z=10$ and $\pi_2: -3x+y-2z=8$.
Find the intersection of the two planes.

5. Let the following two planes be given:
$\pi: -5x+y+8z=-9$ and $\sigma: 2x+5y+8z=9$.
Find the intersection of the two planes.

# 7 Linear transformations

A map $T : \mathbb{R}^m \to \mathbb{R}^n$ assigns elements of a real vector space to elements of another real vector space. This happens in a great variety of applications. In our visual world, many times 3-dimensional reality is mapped into 2 dimensional pictures. If we want this projection to be similar to reality, some properties must hold. Therefore, the following two natural conditions define an important class of maps:

Let $V$ and $W$ be two vector spaces. A **linear transformation** (or linear map) is a function that assigns to each vector $\mathbf{v} \in V$ a unique vector $T(\mathbf{v}) \in W$ and satisfies the following two conditions:

$$\text{(additive) } T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \text{ and (homogeneous) } T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$$

In this course, we only use finite dimensional real vector spaces as $V$ and $W$. That is a clear simplification. In case of $V = W$, linear maps are also called **linear operators**.

**Example 7.1.** *Let $A : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as follows: $A(v_1, v_2) = (v_1 + v_2, v_1 - v_2, 3v_2)$. For instance, $A(2, -3) = (-1, 5, -9)$. Is A linear?*

**Solution:** Let us check the two defining properties to see whether $A$ is linear or not:

On one hand, $A(\mathbf{x}+\mathbf{y}) = A(x_1+y_1, x_2+y_2) = (x_1+y_1+x_2+y_2, (x_1+y_1)-(x_2+y_2), 3(x_2+y_2))$, on the other hand, $A(\mathbf{x}) + A(\mathbf{y}) = (x_1 + x_2, x_1 - x_2, 3x_2) + (y_1 + y_2, y_1 - y_2, 3y_2) = (x_1 + x_2 + y_1 + y_2, x_1 - x_2 + y_1 - y_2, 3(x_2 + y_2))$. We deduce that the additive property is satisfied.

For the homogeneous property, we first calculate $A(\lambda\mathbf{x}) = (\lambda x_1 + \lambda x_2, \lambda x_1 - \lambda x_2, 3\lambda x_2)$ and compare it to $\lambda A(\mathbf{x}) = \lambda(x_1 + x_2, x_1 - x_2, 3x_2) = (\lambda x_1 + \lambda x_2, \lambda x_1 - \lambda x_2, 3\lambda x_2)$. We check that these are equal too.

Therefore $A$ is a linear map. □

In the next example, we show that not every map that "looks" linear is indeed linear.

**Example 7.2.** *Let $B : \mathbb{R} \to \mathbb{R}$ be defined by $B(x) = (2x + 1)$. Is $B$ a linear map?*

**Solution:** Let us check the additive property:

On one hand, $B(x + y) = (2(x + y) + 1)$, on the other hand $B(x) + B(y) = (2x + 1) + (2y + 1) = (2(x + y) + 2)$. They are not equal, hence $B$ is not linear. We do not need to check the other property this time. □

There are two important vector spaces associated to a linear transformation.

The **kernel** of a linear map $T : V \to W$ is the set of vectors in $V$ that are mapped to the zero vector: $ker(T) = \{\mathbf{x} \in V : T(\mathbf{x}) = \mathbf{0})\}$. These vectors form a vector space, so $ker(T)$ has a dimension. Notice that $\mathbf{0}$ is always in the kernel. If this is the only vector, then $dim(ker(T)) = 0$. Usually, we are interested in the non-zero vectors of the kernel.

The **range** of a linear map $T : V \to W$ is the set of vectors in $W$ that we get as images, when we apply $T$ to all vectors of $V$:
$Im(T) = \{\mathbf{y} \in W : \exists\ \mathbf{x} \in V \text{ such that } T(\mathbf{x}) = \mathbf{y}\}$. The vectors of the range form a vector space. The dimension of $Im(T)$ is called the **rank** of $T$.

We recall that every vector space has a basis. The two defining properties of a linear transformation allow us to calculate the image of any vector using the images of the basis vectors. Let $T : V \to W$ be a linear map and $B = \{\mathbf{b_1}, \ldots, \mathbf{b_m}\}$ a basis of $V$. We calculate the images of the basis vectors: $T(\mathbf{b_1}), \ldots, T(\mathbf{b_m})$. We know that any vector $\mathbf{v} \in V$ can be written as a linear combination of the basis vectors. Let $\mathbf{v} = \alpha_1\mathbf{b_1} + \cdots + \alpha_m\mathbf{b_m}$. Using the two properties of a linear map, we get the image of $\mathbf{v}$: $T(\mathbf{v}) = \alpha_1 T(\mathbf{b_1}) + \cdots + \alpha_m T(\mathbf{b_m})$. Since we only consider $V$ and $W$ to be finite dimensional real vectorspaces, this is great news for us. The elements of the standard basis are very simple, it is easy to calculate their images.

Let us collect the images of the basis vectors as columns of a matrix. This matrix is called the **transformation matrix**. $M(T) = (T(\mathbf{b_1}) \ldots T(\mathbf{b_m}))$. Notice the following: if $T$ mapped $\mathbb{R}^m$ to $\mathbb{R}^n$, then the vectors $T(\mathbf{b_i})$ has $n$ components. Therefore $M(T)$ has $n$ rows and $m$ columns. That is, $M(T)$ is of type $n \times m$.

**Theorem 7.3.** *Every linear map $T$ can be written as a matrix multiplication: $T(\mathbf{x}) = M(T)\mathbf{x}$.*

**Example 7.4.** *Let $A : \mathbb{R}^3 \to \mathbb{R}^4$ be the map $(x_1, x_2, x_3) \to (x_1 - x_2, x_2 + x_3, 2x_1 - x_2 - x_3, -x_1 + x_2 + 2x_3)$. Find $M(A)$ and calculate $ker(A)$, $Im(A)$.*

**Solution:** We calculate the image of each unit vector in the standard basis and write them as column vectors to get

$M(A) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 2 \end{pmatrix}$. To find the vectors of the kernel, we have to solve the following system of linear equations: $M(T)\mathbf{x} = \mathbf{0}$. This we can do, for instance, using the Gauss-Jordan elimination.

$$\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

We can deduce that the only solution to the homogeneous equation $M(T)\mathbf{x} = \mathbf{0}$ is the zero vector, the rank of the matrix is 3, the three column vectors are linearly independent. Therefore, $ker(A) = \{\mathbf{0}\}$, the range $Im(A) = \{\alpha \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R}\}$. $\square$

A linear map $T$ is **invertible** (injective) if its transformation matrix $M(T)$ is invertible. This is equivalent to the fact that $det(M(T)) \neq 0$. In that case, the inverse map $T^{-1}$, defined by its transformation matrix $M(T)^{-1}$, is also linear.

In applications, it often happens that we are required to do several linear transformations after each other. Therefore, we next describe the composition of two linear maps. Let $A$ and $B$ be two linear maps. The composition map $A \circ B$ is defined by $A(B(\mathbf{x}))$. We see that it is necessary that $Im(B) \subseteq Dom(A)$. In particular if $B : \mathbb{R}^k \rightarrow \mathbb{R}^m$, then $A$ must map $\mathbb{R}^m$ to $\mathbb{R}^n$. Therefore $M(B)$ has type $m \times k$ and $M(A)$ has type $n \times m$. How can we express $M(A \circ B)$? After seeing the type of $M(A)$ and $M(B)$ it comes as no surprise that $M(A \circ B) = M(A)M(B)$, so $A \circ B(\mathbf{x}) = M(A)M(B)\mathbf{x}$.

# Excercises

1. Let us determine the map, which assigns to any vector of $\mathbb{R}^3$ its orthogonal projection to the $x - y$ plane. Prove that this map is linear. Determine the kernel, the range and the transformation matrix!

2. Consider the following transformations!

   $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(x_1, x_2, x_3) \mapsto (2x_1 + 3x_2, x_1 + x_2 - 3x_3)$.

   $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $(x_1, x_2) \mapsto (x_1 x_2, 4x_1 + x_2^4)$.

   $C : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $(x_1, x_2) \mapsto (3x_1 + 5x_2, 0, x_1 + x_2)$.

   Which one of the above transformations is linear? If it is, calculate its transformation matrix!

3. What is the map, if its transformation matrix is the following?

   $$M(A) = \begin{bmatrix} 5 & 0 & -4 & 2 \\ 1 & 1 & -1 & -5 \end{bmatrix}, \quad M(B) = \begin{bmatrix} -5 & -1 \\ 0 & -1 \\ 5 & 1 \end{bmatrix}, \quad M(C) = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 1 & -2 \\ 3 & 0 & -2 \end{bmatrix}.$$

4. Consider the following linear maps!

   $A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $(x_1, x_2, x_3) \mapsto (2x_1 - x_2 + 4x_3, x_1 + 3x_2 + 2x_3)$.

   $B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $(x_1, x_2, x_3) \mapsto (x_1 + 3x_3, 4x_2, 5x_2 + x_3)$.

   (a) Calculate the transformation matrix of the above maps!

   (b) Let $\mathbf{x} = (2, -1, 3)$. What are the images $A(\mathbf{x})$ and $B(\mathbf{x})$?

   (c) Which one of $A \circ B$ and $B \circ A$ exists? Calculate the transformation matrix for the existing one!

5. Consider the following linear maps!

$A : \mathbb{R}^3 \to \mathbb{R}^2$, $(x_1, x_2, x_3) \mapsto (x_1 - 2x_2 + x_3, x_1 + x_2 + 2x_3)$, $\quad \mathbf{b} = (4, 5)$.

$B : \mathbb{R}^3 \to \mathbb{R}^3$, $(x_1, x_2, x_3) \mapsto (x_1 + 2x_3, 2x_1 + x_2 + x_3, 4x_1 + x_2 + 5x_3)$, $\quad \mathbf{b} = (3, 4, 6)$.

$C : \mathbb{R}^4 \to \mathbb{R}^3$, $(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_3 + x_4, x_1 + 2x_2 + 3x_3 + 5x_4, x_1 + x_2 + 2x_3 + 3x_4)$, $\quad \mathbf{b} = (2, 4, 3)$.

(a) Determine the kernel of each transformation! Which transformation is invertible?

(b) Does the given vector **b** belongs to the range of the transformation? If it does, determine the vectors, whose image is **b**.

# 8 Eigenvalues, eigenvectors

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. It is important to know vectors of the space that are mapped to a parallel vector by a linear operator. A number $\lambda \in \mathbb{R}$ is an *eigenvalue* of $A$, if there exists a non-zero $n$-vector $\underline{v}$ such that $A(\underline{v}) = \lambda\underline{v}$. In this case, $\underline{v}$ is an *eigenvector* that belongs to $\lambda$. We can collect all eigenvectors belonging to the same eigenvalue $\lambda$ in a set $H(\lambda)$. That is, $H(\lambda) = \{\underline{v} \in \mathbb{R}^n : A(\underline{v}) = \lambda\underline{v}\}$. Notice that here the zero vector of $\mathbb{R}^n$ is also included. It is not difficult to prove that $H(\lambda)$ is a vector space called the eigenspace of $\lambda$. The geometric multiplicity of $\lambda$ is simply the dimension of $H(\lambda)$.

Another property that we can deduce from the definitions is that eigenvectors that belong to different eigenvalues are linearly independent. It implies that a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ can have at most $n$ different eigenvalues.

As we learnt in the previous Section, every linear transformation can be represented by a matrix multiplication. Therefore, it is immediate that we can define eigenvalues and eigenvectors of a matrix as follows:

Let $M$ be an $n \times n$ matrix. The number $\lambda \in \mathbb{R}$ is an *eigenvalue* of $M$ if and only if there exists a non-zero $n$-vector $\underline{v}$ such that $M\underline{v} = \lambda\underline{v}$. In that case $\underline{v}$ is an *eigenvector* that belongs to $\lambda$.

Observe that $M\underline{v} = \lambda\underline{v}$ can be written as $M\underline{v} - \lambda\underline{v} = (M - \lambda I_n)\underline{v} = 0$. We deduce the following equivalences buildt on our previous theorems: The vector $\underline{v}$ is an eigenvector belonging to $\lambda$ if and only if the homogeneous system of linear equations $(M - \lambda I_n)\underline{x} = \underline{0}$ has a non-zero solution $\underline{v}$. This latter happens if and only if $\det(M - \lambda I_n) = 0$.

It follows from the definition of the determinant that $\det(M - \lambda I_n)$ is a polynomial in $\lambda$ of degree at most $n$. Therefore, if $A$ is an $n \times n$ matrix, then $p(\lambda) = \det(A - \lambda I_n)$ is the *characteristic polynomial* of $A$ and $p(\lambda) = 0$ is the *characteristic equation.* The roots of the characteristic equation are the eigenvalues of $A$. If the root $\lambda_i$ is a multiple root of multiplicity $m$, then the *algebraic multiplicity* of $\lambda_i$ is $m$. It follows from the Fundamental Theorem of Algebra that $p(\lambda)$ can have at most $n$ real roots. Another fact is that the geometric multiplicity of an eigenvalue is always smaller than or equal to the algebraic multiplicity.

Building on the results of this Section, we can use the following recipe to determine the eigenspaces of a given linear operator in $\mathbb{R}^n$.

1. We determine the transformation matrix $A$.
2. We write the characteristic equation $\det(A - \lambda I_n) = 0$.
3. Solving $p(\lambda) = 0$, we get the eigenvalues $\lambda_1, \ldots$ and their algebraic multiplicities.
4. We substitute each eigenvalue $\lambda_i$ separately into the matrix $A - \lambda_i I_n$.
5. We solve the homogeneous system of linear equations $(A - \lambda_i I_n)\underline{x} = \underline{0}$. The solutions are the eigenvectors in $H(\lambda_i)$. Its dimension is the geometric multiplicity of $\lambda_i$.

**Example 8.1.** *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (x_1 + 3x_2, 2x_1 - 4x_2)$. Determine the eigenvalues and eigenspaces of A. Calculate the algebraic and geometric multiplicity of each eigenvalue.*

**Solution:** We determine the transformation matrix by calculating the images of the standard basis vectors of $\mathbb{R}^2$. We write the images as column vectors of the transformation matrix.

$A = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$.

Next we write the characteristic polynomial: $\det(A - \lambda I_n) = (1 - \lambda)(-4 - \lambda) - 2 * 3 = \lambda^2 + 3\lambda - 10$. We have to find the roots of this polynomial. Since $\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$, the roots are $-5$ and 2. We deduce the algebraic multiplicity of each eigenvalue is 1.

Next we substitute $-5$ in place of $\lambda$ in $A - \lambda I_n$ to get $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}$. Therefore we want the non-trivial solutions of the system $\begin{pmatrix} 6 & 3 \\ 2 & 1 \end{pmatrix}\underline{x} = \underline{0}$. We see that $2x_1 + x_2 = 0$ must hold. Therefore, $H(-5) = \{(x_1, x_2) : x_2 = -2x_1, x_1 \in \mathbb{R}\}$. We deduce the dimension of $H(-5)$ is 1, since there was 1 free variable. This is the geometric multiplicity of the eigenvalue $-5$.

Now we substitute 2 in place of $\lambda$ in $A - \lambda I_n$ to get $\begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix}$. Therefore we want the non-trivial solutions of the system $\begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix}\underline{x} = \underline{0}$. We see that $-x_1 + 3x_3 = 0$ must hold. Therefore,

$H(2) = \{(x_1, x_2) : x_1 = 3x_2, x_2 \in \mathbb{R}\}$. We deduce the dimension of $H(2)$ is 1, since there was 1 free variable. This is the geometric multiplicity of the eigenvalue 2. $\square$

## Excercises

1. Determine the eigenvalues and eigenspaces of the following linear transformation! Calculate the algebraic and geometric multiplicity of each eigenvalue.

   $A : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (4x_1 + 2x_2, 3x_1 - x_2)$.

2. Determine the eigenvalues and eigenspaces of the following linear transformation! Calculate the algebraic and geometric multiplicity of each eigenvalue.

   $A : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (5x_1 - 2x_2, -4x_1 + 3x_2)$.

3. Determine the eigenvalues and eigenspaces of the following linear transformation! Calculate the algebraic and geometric multiplicity of each eigenvalue.

   $A : \mathbb{R}^2 \to \mathbb{R}^2$, $(x_1, x_2) \mapsto (-3x_1 + 2x_2, -4x_1 + 6x_2)$.

4. Determine the eigenvalues and eigenspaces of the following linear transformation! Calculate the algebraic and geometric multiplicity of each eigenvalue.

   $A : \mathbb{R}^3 \to \mathbb{R}^3$, $(x_1, x_2, x_3) \mapsto (3x_1, -x_1 + x_2, -3x_1 - 2x_2 + 3x_3)$.

5. True or false?

   A linear operator can have an eigenvalue for which there is only one associated eigenvector.

   A linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ can have at most $n$ different eigenvectors.

## Solutions

**Section 1.2**

1. $x_1 = 4$, $x_2 = -3$, $x_3 = 2$.
2. $x_1 = 1$, $x_2 = -1$, $x_3 = 0$, $x_4 = 2$.
3. $x_1 = 2$, $x_2 = 3$, $x_3 = -1$.
4. $x_1 = -\frac{3}{2}$, $x_2 = \frac{17}{3}$, $x_3 = \frac{5}{18}$
5. $x_1 = 7 - 2x_2 + x_3 - x_4$, $x_2, x_3, x_4 \in \mathbb{R}$.

**Section 2.1**

1. $(-77, -87, -39, 1)$ and $(-21, 21, -49, -56)$.
2. $(8, 21, -23)$.
3. $|\mathbf{a}| = \sqrt{9 + 16 + 25 + 81} = \sqrt{131}$.

**Section 2.2**

1. $37$, $-24$, $-1$.
2. $xy + yz + zx$.
3. Since $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + \cdots + a_k^2$, it must be non-negative.
4. $294$, $15$, $-694$, $-375$.

**Section 2.3**

1.(b) $(-6, -11, 11)$.

(c) $|\underline{u}| = \sqrt{14}$, $|\underline{v}| = \sqrt{17}$.

(d) We know that $\cos\alpha = \dfrac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \dfrac{-7}{\sqrt{238}}$. Hence $\alpha = \arccos \frac{-7}{\sqrt{238}}$.

(e) $-\underline{v} = (0, 1, -4)$, for instance $3\underline{v} = (0, -3, 12)$ is parallel and $(10, 4, 1)$ is perpendicular to $\underline{v}$, since their scalar product is zero.

(f) $\frac{\underline{v}}{|\underline{v}|} = (0, \frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$.

(g) $4(0, \frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$ and $\frac{1}{3}(0, \frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}})$.

2.(a) $\text{proj}_{\underline{a}}\underline{v} = \dfrac{\underline{v} \cdot \underline{a}}{|\underline{a}|^2}\underline{a} = \dfrac{28}{14}(2, -1, 3) = (4, -2, 6)$.

(b) $\underline{v} = (4, -2, 6) + (0, 9, 3)$.

**Section 3**

1. Yes, $\mathbf{c} = 4\mathbf{b} - \mathbf{a}$.
2. No.
3. Only with the trivial combination $0\mathbf{a} + 0\mathbf{b} + 0\mathbf{c}$. Therefore $H$ is linearly independent.
4. Yes. The coordinates of $\mathbf{v}$ with respect to $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ are $1, 2, 3$.
5. Only with the trivial combination $0\mathbf{a} + 0\mathbf{b} + 0\mathbf{c}$. Therefore $H$ is linearly independent. No.
6. The two vectors of $H_1$ are independent. Any basis of $\mathbb{R}^3$ has 3 elements. Therefore, $H_1$ is not a basis and not a generating set either.

   The vectors of $H_2$ are independent, they form a basis and therefore a generating set as well.

   Since $H_3$ contains $H_2$, therefore $H_3$ is a generating set, but not a basis. Therefore $H_3$ is dependent.
7. The rank $r(H) = 2$. Use the linear combination of the two new basis vectors to add a vector that is dependent on them.
8. Hint: Show, using the basis transformation process, that the union of the basis of $V_1$ and $V_2$ is a basis of $\mathbb{R}^3$. $v_1 = (4, 4, 4)$ and $v_2 = (-3, 6, -2)$.

**Section 4**

1. $$\begin{pmatrix} -2 & -5 & -8 & -11 & -14 \\ 1 & -2 & -5 & -8 & -11 \\ 6 & 3 & 0 & -3 & -6 \\ 13 & 10 & 7 & 4 & 1 \\ 22 & 19 & 16 & 13 & 10 \end{pmatrix}$$

2. $\begin{pmatrix} -7 & -11 & -15 & -19 & -23 & -27 \\ -10 & -14 & -18 & -22 & -26 & -30 \\ -13 & -17 & -21 & -25 & -29 & -33 \\ -16 & -20 & -24 & -28 & -32 & -36 \\ -19 & -23 & -27 & -31 & -35 & -39 \\ -22 & -26 & -30 & -34 & -38 & -42 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{pmatrix}$

4. $A^T = \begin{pmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 0 & 3 & 6 \\ -1 & 2 & 5 \end{pmatrix}$

**Section 4.1**

1. $\begin{pmatrix} 33 & -96 & 102 \\ 36 & -30 & -39 \\ 105 & -6 & 24 \end{pmatrix}$

2. $\begin{pmatrix} 10 & -4 & 7 \\ 17 & -15 & 22 \end{pmatrix}$

**Section 4.2**

1. $\begin{pmatrix} 11 & 16 \end{pmatrix}$

2. $AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $AC = A$, $CA = C$. We see that the zero matrix can have a divisor. A matrix $X$, other than the identity can have the property that $AX = A$.

3. $AB = \begin{pmatrix} 26 & 29 & 32 & 35 \\ 58 & 65 & 72 & 79 \\ 30 & 35 & 40 & 45 \end{pmatrix}$, $CB^T = \begin{pmatrix} -130 & -202 \\ -10 & -18 \\ 110 & 166 \end{pmatrix}$.
$BA$ and $BC$ do not exist.

4. $CA = \begin{pmatrix} 1 & 1 & 3 \\ 6 & 0 & 9 \\ 6 & -2 & 6 \\ 2 & -4 & -3 \end{pmatrix}$, $AB = \begin{pmatrix} 1 & 5 & -5 \\ 13 & 2 & 6 \end{pmatrix}$, $BA^T = \begin{pmatrix} -2 & -5 \\ 2 & 8 \\ 5 & 18 \end{pmatrix}$,
$AC$, $A^TB^T$, $C^TB$ do not exist.

5. Matrix multiplication is associative. $A(BC) = (AB)C = \begin{pmatrix} -179 \end{pmatrix}$.

6. $AB = \begin{pmatrix} -7 & -4 & -11 \\ 6 & 2 & 10 \end{pmatrix}$, $BC = \begin{pmatrix} -24 & -7 & 24 \\ -21 & -3 & 35 \end{pmatrix}$, $A(BC) = (AB)C = \begin{pmatrix} 39 & 2 & -81 \\ -42 & -6 & 70 \end{pmatrix}$.

7. The determinant is 0, hence $A$ inverse does not exist.

8. $A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ 7 & -6 & 1 \end{pmatrix}$

9. $A^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$

10. $B^{-1} = \frac{1}{19}\begin{pmatrix} 11 & -3 & -5 \\ -14 & 9 & 15 \\ 10 & -1 & -8 \end{pmatrix}$

11. We only define inverse of a square matrix, so $A^T$ does not have an inverse.

$B^{-1} = \frac{1}{8}\begin{pmatrix} 3 & 1 & -2 \\ -1 & -3 & 6 \\ -2 & 2 & 4 \end{pmatrix}$

**Section 5**

1. Positive, positive.

2. In the calculation of a $5 \times 5$ determinant, we have to multiply 5 terms in each summand such that the terms lie in different rows and columns. In our case, there are two rows and two columns consisting of non-zero elements. If we select 5 items, one in each row and column, then one of them must be a 0. Therefore each summand is 0, hence the determinant is 0 as well.

3. In a $4 \times 4$ determinant, we have to calculate (4! times) the product of 4 entries that lie in different columns and rows. In our case, the determinant will be a polynomial of degree at most 4. There is only one quadruple of entries such that each of them contains $x$. Therefore, the coefficient of $x^4$ is 2.

Notice the following: If we select a number from the matrix, then the $x$'s in its row and column cannot be used for the product of four entries. Therefore, the only way to get a product that contains three $x$'s is to select the 1 in the second row and first column. Now, we have to decide its sign! It is negative. So the coefficient of $x^3$ is -1.

4. Use the elementary row operations for matrices! Try to eliminate like in Gauss-Jordan and factor out common terms.

5. Same as in the previous exercise.

6. -18, 0, 160, 48, 1, -33, 230.

7. $abc - ab + a - 1$.

8. -696, 62, -174.

9. $2a - b - c - d$.

**Section 5.1**

1. $x_1 = 2$, $x_2 = 5$, $x_3 = -3$.

2. $x_1 = 0$, $x_0 = 0$, $x_3 = 1$.

3. $x_1 = 1$, $x_2 = 2$, $x_3 = -1$, $x_4 = -2$.

4. $x_1 = -2$, $x_2 = 2$, $x_3 = -3$, $x_4 = 3$.

**Section 6**

1. The parametric equations are $x = 3 + 4t$, $y = 1 + 5t$, $z = -4$. The symmetric equations are: $\frac{x-3}{4} = \frac{y-1}{5}$ and $z = -4$.

2. The parametric equations are $x = 1 + 2t$, $y = 4 + 2t$, $z = 5 - 6t$. The symmetric equations are: $\frac{x-1}{2} = \frac{y-4}{2} = \frac{z-5}{-6}$.

3. For the line $e$: we find a point $P = (2, -1, 5)$ and the direction vector $v = (3, 2, -4)$. The symmetric equations are $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z-5}{-4}$.

For the line $f$: we find a point $P = (0, -2, 4)$ and the direction vector $v = (5, 7, 0)$. The symmetric equations are $\frac{x}{5} = \frac{y+2}{7}$ and $z = 4$.

For the line $g$: we find a point $P = (6, 1, 0)$ and the direction vector $v = (0, 3, 0)$. There is no symmetric equation and the line $x = 6$, $z = 0$ is parallel to the $y$-axis.

4. The parametric equation is: $x = 3 - 5t$, $y = 4$ and $z = -1 + 7t$.

5. The parametric equation is: $x = 1 + 2t$, $y = -1 + 3t$ and $z = 4 + t$.

6. The normal vector is $(2, -3, 5)$. For instance the following points belong to $\pi$: $(1, 1, 6/5)$, $(0, 0, 1)$, $(1, -1, 0)$. We see by substitution that $(-8, 3, 6)$ belongs to the plane, and $(1, 4, -3)$ does not.

7. The equation of the plane is: $2x + 3y - z + 3 = 0$.

8. The equation of the plane is: $2x + y + 3z - 9 = 0$.

9. The equation of the plane is: $-30x + 76y - 50z + 292 = 0$.

10. The equation of the plane is: $31x - 59y - 6z = 101$.

**Section 6.1**

1. The lines $e$ and $f$ intersect each other in point $(0, 2, -2)$.
The lines $e$ and $g$, and similarly the lines $f$ and $g$ do not intersect.

2. They intersect in a point $(-3, -3, 0)$.

3. They intersect in a point $(-9, -18, -1)$.

4. The two planes have a common point, for instance $(6, 5, 2/3)$ is a solution of both equations. We get the direction of the intersection line using the vector product: $(2, -5, 1) \times (-3, 1, -2) = (9, 1, -13)$. Therefore the parametric equations of the intersection line is the following: $x = 6 + 9t$, $y = 5 + t$ and $z = 2/3 - 13t$.

5. The two planes have a common point, for instance $(2, 1, 0)$. We get the direction of the intersection line using the vector product: $(-5, 1, 8) \times (2, 5, 8) = (-32, 56, -27)$. Therefore the parametric equations of the intersection line is the following: $x = 2 - 32t$, $y = 1 + 56t$ and $z = -27t$.

**Section 7**

1. The map is $A : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (x, y, 0)$. It is easy to check the additive and homogeneous property.

The kernel consists of vectors with first two components being zero. $ker(A) = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0, z \in \mathbb{R}\}$. It has dimension 1.

Any vector in the $x - y$ coordinate plane can be the image of some vector in 3D. Therefore, $Im(A) = \{\mathbf{v} \in \mathbb{R}^3 : v_3 = 0\}$.

The transformation matrix is $M(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

2. The map $A$ is linear. Its transformation matrix is $M(A) = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & -3 \end{pmatrix}$.

The map $B$ is non-linear.

The map $C$ is linear again. Its transformation matrix is $M(C) = \begin{pmatrix} 3 & 5 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

3. $A : \mathbb{R}^4 \to \mathbb{R}^2$, $(x_1, x_2, x_3, x_4) \mapsto (5x_1 - 4x_3 + 2x_4, x_1 + x_2 - x_3 - 5x_4)$.
$B : \mathbb{R}^2 \to \mathbb{R}^3$, $(x_1, x_2) \mapsto (-5x_1 - x_2, -x_2, 5x_1 + x_2)$.
$C : \mathbb{R}^3 \to \mathbb{R}^3$, $(x_1, x_2, x_3 \mapsto (3x_1 - x_2, 2x_1 + x_2 - 2x_3, 3x_1 - 2x_3)$.

4.(a) $M(A) = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 2 \end{pmatrix}$, $\quad M(B) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 5 & 1 \end{pmatrix}$.

(b) $A(x) = (17, 5)$ and $B(x) = (11, -4, -2)$.

(c) Only $B \circ A$ exists. Its transformation matrix is $M(A)M(B) = \begin{pmatrix} 2 & 16 & 10 \\ 1 & 22 & 5 \end{pmatrix}$.

5.(a) $ker(A) = \{(x_1, x_2, x_3) : x_1 = 5x_2, x_3 = -3x_2, x_2 \in \mathbb{R}\}$, so $A$ is not invertible.
$ker(B) = \{(x_1, x_2, x_3) : x_1 = -2x_3, x_2 = 3x_3, x_3 \in \mathbb{R}\}$, so $B$ is not invertible.
$ker(C) = \{(x_1, x_2, x_3, x_4) : x_1 = -x_3 - x_4, x_2 = -x_3 - 2x_4, x_3, x_4 \in \mathbb{R})\}$, so $C$ is not invertible.

(b) For $A$, we get that vectors $(x_1, x_2, x_3)$ satisfying $x_1 = 3 + 5x_2$ and $x_3 = 1 - 3x_2$ are mapped to $(4, 5)$.

For $B$, we get a contradiction using any method solving the equations, so $(3, 4, 6)$ does not belong to the range.

For $C$, we get that vectors $(x_1, x_2, x_3, x_4)$ satisfying $x_1 = 2 - x_3 - x_4$ and $x_2 = 1 - x_3 - 2x_4$ are mapped to $(2, 4, 3)$.

**Section 8**

1. The eigenvalues are 5 and $-2$, both have algebraic multiplicity 1. The eigenspace $H(5) = \{(x_1, x_2) : x_1 = 2x_2, \quad x_2 \in \mathbb{R}\}$. The eigenspace $H(-2) = \{(x_1, x_2) : x_2 = -3x_1, \quad x_1 \in \mathbb{R}\}$. Therefore both eigenvalues have geometric multiplicity 1.

2. The eigenvalues are 1 and 7, both have algebraic multiplicity 1. The eigenspace $H(1) = \{(x_1, x_2) : x_2 = 2x_1, \quad x_1 \in \mathbb{R}\}$. The eigenspace $H(7) = \{(x_1, x_2) : x_2 = -x_1, \quad x_1 \in \mathbb{R}\}$. Therefore both eigenvalues have geometric multiplicity 1.

3. The eigenvalues are 5 and $-2$, both have algebraic multiplicity 1. The eigenspace $H(5) = \{(x_1, x_2) : x_2 = 4x_1, \quad x_1 \in \mathbb{R}\}$. The eigenspace $H(-2) = \{(x_1, x_2) : x_1 = 2x_2, \quad x_2 \in \mathbb{R}\}$. Therefore both eigenvalues have geometric multiplicity 1.

4. The eigenvalues are 1 and 3. The eigenvalue 3 has algebraic multiplicity 2, the eigenvalue 1 has algebraic multiplicity 1. The eigenspace $H(1) = \{(x_1, x_2, x_3) : x_1 = 0, x_3 = x_2, \quad x_2 \in \mathbb{R}\}$. The eigenspace $H(3) = \{(x_1, x_2, x_3) : x_1 = x_2 = 0, \quad x_3 \in \mathbb{R}\}$. Therefore both eigenvalues have geometric multiplicity 1.

5. False, if there is one eigenvector, then all its scalar multiples are eigenvectors too.

False, this is true for eigenvalues not for eigenvectors.