Problems and Applications of Operations Research

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Contents

Notations

k, t	normal lower case letters: scalar quantities
$i, \ldots, n$	(generally) integer quantities
$\mathbf{v}$	bold face, lower case: (column) vector
v_i	i-th component of vector $\mathbf{v}$
$\mathbf{v}^T$	transpose of $\mathbf{v}$: a row vector
$\mathbf{A}$	bold face, capital: matrix
$\mathbf{a}^i$	row i of matrix $\mathbf{A}$ (row vector!)
$\mathbf{a}_j$	column j of matrix $\mathbf{A}$
a^i_j	entry of matrix $\mathbf{A}$ at the intersection of row i and column j
m	(generally) number of rows in a matrix or the dimension of a vector
n	(generally) the number of columns in a matrix
$\mathbb{R}^m$	m dimensional Euclidean space of real numbers
$\mathbb{R}^{m \times n}$	$m \times n$ dimensional Euclidean space of real numbers
$\mathcal{F}$	calligraphic capital: set

Introduction

When learning a new discipline it is equally important to study the theoretical foundations and also to practice the newly acquired knowledge by solving relevant problems. The purpose of this collection is the assist the latter.

Operations research is a relatively new discipline. Sometimes it is referred to as the science of decision making. As such, it is really application oriented though the theoretical foundations are based on unique mathematical techniques and algorithms. There exist several textbooks on operations research even in Hungarian. Recently a new tendency has emerged to help students in their studies by providing teaching material in English. Publishing the current notes belongs to this trend.

The approach of these notes is different to other ones. This set is particularly meant for advanced studies. In order to help the users some theoretical background is provided at the front of some chapters in Part I, mostly for the simplex method of linear programming as we deal with advanced solution algorithms. Volume restrictions do not make it possible to provide extensive material and the number of exercises is also limited, while the scope of operations research is rather wide. Thus, this collection is the result of some compromises. However, we believe, even with these restrictions it can be of great help to interested parties.

Users of these notes are assumed to have some prior knowledge of the theoretical background of relevant areas, mostly in linear algebra and calculus. Successful completion of a introductory courses in optimization and, in particular, game theory are also necessary.

The main areas covered by these notes are based on the most commonly used algorithms in operations research. Naturally, there are several areas out of the scope of these notes but the successful completion of these exercises yields a well-founded knowledge for the learners. The principles covered here can enhance the understanding of other books and publications. Generally these notes can help logistic students to improve their knowledge, since transportation oriented problems are highlighted in a separate section.

These notes are divided into two parts. Part I contains the problems by subject areas. In some cases a summary of the relevant theoretical background is provided. In Part II the same structure is used. The description of problems is repeated here for easier reference followed by the solution. Typically, for the first (or first few) problems model answers are given so that the user can check the process of the solution. It enables him/her to solve similar problems without any difficulty.

The notes are not connected to any textbook. This is why some theory is also given before the exercises are provided. Where such a summary is missing the user is supposed to be knowledgeable about the topic or should do some exploratory work.

Current version of the notes will regularly be extended by adding new exercises and, maybe, including more chapters of operations research. The version number of the notes will change accordingly.

Veszpr July 2013

The Authors

Part I

Problems and Exercises

Chapter 1

Basics of background of operations research

1.1 Linear algebra

1.1.1 Exercises

1. What does triangle inequality say for the norms of two m dimensional vectors $\mathbf{a}$ and $\mathbf{b}$?

2. Which of the following pairs of vectors are orthogonal? Why?

   (*a*) $[1, 2]$ and $[-1, 1]$,
   (*b*) $[2, 5, 1]$ and $[-3, 1, 1]$,
   (*c*) $[0, 1, -1.98]$ and $[1, 0.99, 1/2]$,
   (*d*) $[3, 5, 3, -4]$ and $[4, -2, 2, 2]$.

3. Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

   (*a*) $\mathbf{b} = [4, 5]$, $\mathbf{a}_1 = [1, 3]^T$ and $\mathbf{a}_2 = [2, 2]^T$,
   (*b*) $\mathbf{b} = [1, -2]$, $\mathbf{a}_1 = [2, 1]^T$ and $\mathbf{a}_2 = [5, 5]^T$,
   (*c*) $\mathbf{b} = [1, -2]$, $\mathbf{a}_1 = [2, -3]^T$ and $\mathbf{a}_2 = [2, -8]^T$,
   (*d*) $\mathbf{b} = [2, -15]$, $\mathbf{a}_1 = [3, -4]^T$ and $\mathbf{a}_2 = [14, 6]^T$.

4. Which of the following sets of vectors are linearly independent:

   (*i*) $[1, 5]$, $[2, 3]$;
   (*ii*) $[2, 1, -3]$, $[-1, 1, -6]$, $[1, 1, -4]$.

5. Show that vectors $\mathbf{a}_1 = [2, 3, 1]^T,\ \mathbf{a}_2 = [1, 0, 4]^T,\ \mathbf{a}_3 = [2, 4, 1]^T,\ \mathbf{a}_4 = [0, 3, 2]^T$ are linearly dependent.

6. For each of the following statements, determine whether it is true or false. Justify your answer.

   (*i*) A basis must contain $\mathbf{0}$.

   (*ii*) Subsets of linearly dependent sets are linearly dependent.

   (*iii*) Subsets of linearly independent sets are linearly independent.

   (*iv*) If $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n = \mathbf{0}$ then all scalars λ_j are zero.

   (*v*) Any set of m vectors containing the null vector is linearly dependent.

   (*vi*) The dot product of two, linearly dependent, nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ is always equal to zero.

   (*vii*) If a matrix is multiplied by a diagonal matrix, the result does not depend on the order of multiplication.

   (*viii*) The product of two square matrices is always defined.

7. Answer the following questions:

   (*a*) How is the rank of an $m \times n$ matrix defined?

   (*b*) What is the relationship between the *row rank* and *column rank* of an $m \times n$ matrix $\mathbf{A}$?

   (*c*) What does *full rank* of an $m \times n$ matrix $\mathbf{A}$ mean?

8. Answer the following questions:

   (*a*) How is the basis of a vector space defined?

   (*b*) What is the size of a basis in $\mathbb{R}^n$? Is a basis unique for $\mathbb{R}^n$?

   (*c*) Which, if any, of the following systems of vectors are bases in $\mathbb{R}^3$:

      (i) $[1, 3, 2], [3, 1, 3], [2, 10, 2]$.

      (ii) $[1, 2, 1], [1, 0, 2], [2, 1, 1]$.

9. Answer the following questions:

   (*a*) How is the p-norm of a vector $\mathbf{v} \in \mathbb{R}^m$ defined? What are the important special cases?

   (*b*) What does the triangle inequality say for the norms of two compatible matrices $\mathbf{A}$ and $\mathbf{B}$?

10. Determine norms $\|\mathbf{A}\|_1$ and $\|\mathbf{A}\|_\infty$ of the given matrix $\mathbf{A}$:

   (*a*)

$$\mathbf{A} = \begin{bmatrix} -9 & 2 & 3 \\ -4 & 8 & 6 \\ 1 & 5 & 7 \end{bmatrix}.$$

(*b*)

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 3 \\ -10 & 2 & 1 \\ 1 & 6 & -8 \end{bmatrix}.$$

(*c*)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & -2 \\ -6 & -4 & 1 \\ 4 & 3 & 9 \end{bmatrix}.$$

(*d*)

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & -2 \\ -6 & 4 & 1 \\ 3 & -3 & 7 \end{bmatrix}.$$

11. Solve the following system of equations using Gauss-Jordan elimination. Identify basic variables. Express all solutions in terms of non-basic variables.

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & - & x_5 & = & -2 \\ 4x_1 & + & 2x_2 & & & + & 3x_4 & - & 2x_5 & = & 2 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \end{array}$$

12. For

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

decide which of the following products are defined, and compute them:

(*a*) $\mathbf{Au}$, (*b*) $\mathbf{Av}$, (*c*) $\mathbf{A}^T\mathbf{v}$, (*d*) $\mathbf{u}^T\mathbf{v}$, (*e*) $\mathbf{uv}^T$.

13. Given matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix},$$

where $a,\ b,\ c$ and d are scalars. Compute $\mathbf{AB} - \mathbf{BA}$.

Give conditions for $\mathbf{AB} = \mathbf{BA}$.

14. Under what conditions are the following matrix equalities true?

(*a*) $(\mathbf{X} + \mathbf{Y})^2 = \mathbf{X}^2 + 2\mathbf{XY} + \mathbf{Y}^2$.

(*b*) $(\mathbf{X} + \mathbf{Y})(\mathbf{X} - \mathbf{Y}) = \mathbf{X}^2 - \mathbf{Y}^2$.

15. Prove the following statements:

(*a*) Show that for any $m \times n$ matrix $\mathbf{A}$, both $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric. Give the dimensions of these matrices.

(*b*) Show that matrix $\mathbf{A}^T\mathbf{A}$ is positive semidefinite.

(*c*) Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be nonsingular matrices. Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

(*d*) Prove that $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$.

16. How is the inverse of a matrix defined? Which matrices have an inverse? What are the main properties of the inverse?

1.2 The linear programming problem

1.2.1 Summary of theoretical background for this section

1.2.1.1 General form of LP problems

There are n **decision variables** $x_1, x_2, \dots, x_n$ called the **structural variables** of the LP problem.

A linear function (called **objective function**) of these variables is to be minimized subject to linear constraints:

$$\min \; z = c_1x_1 + \cdots + c_nx_n \;\text{ or }\; z = \sum_{j=1}^{n} c_jx_j$$

The **general constraints** that involve more than one variable can have lower and upper limits:

$$L_i \le \sum_{j=1}^{n} a_j^i x_j \le U_i, \quad i = 1, \dots, m,$$

where m is the number of such constraints. Individual constraints on variables (**bounds**) in the general form look like:

$$\ell_j \le x_j \le u_j \quad j = 1, \dots, n.$$

Any of the lower bounds (L_i or ℓ_j) can be $-\infty$. Similarly, any of the upper bounds (U_i or u_j) can be $+\infty$.

If $u_j = +\infty$ and ℓ_j is finite x_j is called **plus type variable**, PL.

If $\ell_j = -\infty$ and u_j is finite, then x_j is of **minus type**, MI. An MI variable can be converted into PL by multiplying it by -1.

If both ℓ_j and u_j are finite, i.e., $\ell_j \le x_j \le u_j$, then x_j is called **bounded variable**, BD. Special sub-case: $\ell_j = u_j = x_j$. Such an x_j is called **fixed variable**, FX.

If $\ell_j = -\infty$ and $u_j = +\infty$, i.e., $-\infty \le x_j \le +\infty$, x_j is called unrestricted or **free variable**, FR.

Finite lower bounds on variables can be shifted to zero (**translation**).

If a variable takes a value within its bounds it is said to be at **feasible** level.

Based on the finiteness of the individual bounds of variables it is worth distinguishing 4 types of variables:

Feasibility range	Type	Reference
$x_j = 0$	0	Fixed
$0 \leq x_j \leq u_j$	1	Bounded
$0 \leq x_j \leq +\infty$	2	Nonnegative
$-\infty \leq x_j \leq +\infty$	3	Free

General constraints can also be classified based on the finiteness of L_i and U_i.

If $L_i = -\infty$ and U_i is finite: $\sum_{j=1}^n a_j^i x_j \leq U_i$. It is an LE, **less than or equal to** type constraint that can be converted into equation: $z_i + \sum_{j=1}^n a_j^i x_j = b_i$ with $b_i \equiv U_i$ and $z_i \geq 0$.

If $U_i = +\infty$ and L_i is finite: $L_i \leq \sum_{j=1}^n a_j^i x_j$ which is known as GE or **greater than or equal to** type constraint. Denoting $b_i = -L_i$, the equivalent form: $\sum_{j=1}^n (-a_j^i) x_j \leq b_i$. It can be converted into equation: $z_i + \sum_{j=1}^n (-a_j^i) x_j = b_i$ with $z_i \geq 0$.

If both L_i and U_i are finite then we have two general constraints $L_i \leq \sum_{j=1}^n a_j^i x_j$ and $\sum_{j=1}^n a_j^i x_j \leq U_i$. They are equivalent to a general and an individual constraint: $z_i + \sum_{j=1}^n a_j^i x_j = U_i$ with $0 \leq z_i \leq (U_i - L_i)$. This is called a **range constraint**, RG. Denoting $b_i = U_i$ and $r_i = (U_i - L_i)$: $z_i + \sum_{j=1}^n a_j^i x_j = b_i$ with $0 \leq z_i \leq r_i$. True even if $L_i = U_i$ in which case $z_i = 0$. This is referred to as an **equality constraint**, EQ.

If $L_i = -\infty$ and $U_i = +\infty$ then with arbitrary b_i, formally: $z_i + \sum_{j=1}^n a_j^i x_j = b_i$ with z_i unrestricted (free). This is called a **non-binding constraint**, NB.

With the addition of a proper z_i **logical variable** each constraint becomes an equality. Logical variables fall into the same 4 classes as the structural ones (see above).

By the introduction of logical variables any general constraint can be brought into the form (**computational form** #1, CF #1) of:

$$\begin{array}{ll} \min & z = \displaystyle\sum_{j=1}^n c_j x_j \\ \text{s.t.} & z_i + \displaystyle\sum_{j=1}^n a_j^i x_j = b_i, \text{ for } i = 1, \ldots, m \\ & \text{and the type constraints on } x_j \text{ and } z_i \text{ variables.} \end{array}$$

1.2.2 Exercises

1. Convert the following linear programming constraints into equalities. Indicate the type of the associated logical variable. Try to combine constraints if possible.

$$2x_1 - 3x_2 + 4x_3 - 5x_4 \le 6 \tag{1.1}$$
$$x_1 + x_2 - 3x_3 - x_4 \le -6 \tag{1.2}$$
$$3x_1 + x_3 - x_4 \ge 2 \tag{1.3}$$
$$-2x_2 + 3x_4 \ge -1 \tag{1.4}$$
$$2x_3 + x_4 \le 9 \tag{1.5}$$
$$2x_3 + x_4 \ge 4 \tag{1.6}$$
$$12 \ge 3x_1 - x_2 + x_3 + 2x_4 \ge 5 \tag{1.7}$$
$$x_1 + x_2 + x_3 + x_4 \bowtie 8 \tag{1.8}$$
$$x_1 + x_2 - x_3 - x_4 = 0 \tag{1.9}$$

Symbol $\bowtie$ indicates “nonbinding” (NB) constraints.

2. Convert the following set of LP constraints to computational form #1. Indicate the type of newly introduced variables (if any). Try to combine constraints if possible.

$$x_1 + 2x_3 \le 1 - x_4 \tag{1.10}$$
$$2x_1 + x_2 - 3x_3 - x_4 \le -1 \tag{1.11}$$
$$3x_1 + x_3 - x_4 \ge 0 \tag{1.12}$$
$$x_1 + 2x_3 + x_4 \ge -1 \tag{1.13}$$
$$9 \ge 2x_1 - x_2 + x_3 - 2x_4 \ge -1 \tag{1.14}$$
$$x_1 + x_2 + x_3 + x_4 \bowtie 0 \tag{1.15}$$
$$x_1 + x_2 - x_3 = x_4 \tag{1.16}$$

Symbol $\bowtie$ indicates “nonbinding” (NB) constraints.

3. The following LP problem has two general constraints and four variables:

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - 5x_4 &\le 6 \\ 3x_1 - 4x_2 + 2x_3 + 2x_4 &\ge 5 \end{aligned}$$

$$-1 \le x_1 \le 0,\ x_2 \ge 0,\ x_3 \le -2,\ x_4 \text{ free.}$$

Convert the joint constraint into equalities. Reverse minus type variables, if any, shift all finite lower bounds to zero. Indicate the type of newly created variables.

4. Consider the following linear programming problem:

$$
\begin{array}{rrcrcrcr}
\min & -2x_1 & + & 4x_2 & - & 12x_3 & & \\
\text{s.t.:} & -2x_1 & + & 4x_2 & - & 2x_3 & = & 0 \\
4 \geq & x_1 & - & 3x_2 & + & x_3 & \geq & 12 \\
 & 3x_1 & - & 6x_2 & + & 2x_3 & \geq & 5 \\
 & \multicolumn{7}{c}{x_1 \geq -2,\ x_2 \geq 0,\ \ -1 \leq x_3 \leq 1}
\end{array}
$$

Convert the joint constraint into equalities. Reverse minus type variables, if any, shift all finite lower bounds to zero. Indicate the type of newly created variables. Don't forget to convert the objective function too.

5. What is the approximate number of potentially different bases for an LP problem with 27 constraints ($m = 27$) and 81 variables ($n = 81$).

Hint: use the Stirling formula

$$k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k.$$

Take $\pi = 3.14$ and $e = 2.71$ (calculator needed).

6. A chemical plant can produce 5 different types of fertilizer, F-1, ..., F-5. The production requires labour, energy, and processing on machines. These resources are available in limited amounts. The company wants to determine what quantities to produce that maximize the monthly revenue, assuming that any amount can be sold. The following table describes the technological requirements of producing one unit (tonne) of each product, the corresponding revenue and the monthly availability of the resources.

	F-1	F-2	F-3	F-4	F-5	Limit
Revenue	5	6	7	5	6	
Machine hour	2	3	2	1	1	1050
Labour hour	2	1	3	1	3	1050
Energy	1	2	1	4	1	1080

For instance, to produce one tonne of F-3 1 unit of energy is needed.

Formulate the linear programming model of the problem.

7. A cattle farmer wants to minimize feeding costs while making sure the animals get the necessary weekly quantities of the four main nutrients. They are available in three stocks according to the following table.

	St-1	St-2	St-3	Required
Unit cost	8	9	7	
Nutr-1	4	3	2	600
Nutr-2	1	3	3	550
Nutr-3	2	2	0	400
Nutr-4	4	5	7	800

For instance, one unit of St-2 contains 2 units of nutrient 3. Column "Required" contains the minimum weekly requirements. One additional constraint is that no more than 300 units of St-1 is available per week.

Formulate the linear programming model of the problem.

8. A cargo company is preparing a ship with three stowages: front deck, rear deck and main stowage. Each stowage has a weight limit and a space capacity with the following limits:

Stowage	Weight (t)	Space (m^3)
Front deck	10	10000
Rear deck	6	4500
Main stowage	20	8000

The following four cargoes are waiting to be shipped:

Cargo	Available quantity (t)	Volume (m^3/t)	Profit ($/$t$)
C1	12	480	190
C2	10	550	220
C3	20	390	170
C4	16	600	250

Any proportion of the cargoes can be accepted if they are delivered. Formulate an optimization problem to maximize the profit of the delivery.

9. The HR staff of a hospital would like to calculate the minimal number of nurses required for appropriate operation. The nurses are scheduled weekly in three shifts (6:00-14:00, 14:00-22:00, 22:00-06:00). The hospital needs nurses all the time, the minimum number of nurses required in a working week is given for each shift in the following table:

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Morning	8	9	7	11	8	6	5
Afternoon	9	7	8	10	4	3	3
Night	4	3	3	4	3	2	2

The work schedule of a nurse must meet the following requirements:

- Each nurse is assigned in the same shift (morning, afternoon or night) during a working period.
- The working period of a nurse consists of five consecutive days during any seven day long period.

Formulate an optimization problem that helps the HR staff determine the minimal number of nurses.

10. An electric company runs two coal-fired power plants, a new and an old one. These plants use different technologies. So, burning a ton of coal costs \$20 in the new plant or \$15 in the old plant. They produce 6150 kW/h or 5500 kW/h of electricity, respectively, while burning a ton of coal. There are three coal mines in the area. The monthly available amounts, supplier prices (\$) and the transportation costs (\$) are given in the following table:

Mine	Available amount (t)	Price $\$/t$
Expensive	400	70
Faraway	600	55
Fair	300	60

Also the transportation costs (in \$) for a ton of coal are different depending on the physical locations of the mines and the plants:

	New plant	Old plant
Expensive	8	15
Faraway	30	25
Fair	12	13

Formulate a linear programming problem to maximize the monthly profit of the plants, if the selling price of 1 kW/h power is \$0.1 and each plant has a capacity limit of 500 tons for a month, assuming there is no loss in the produced amounts.

11. A company is considering three new products to replace current ones that are being discontinued. Management wants to determine which mix of these new products should be produced while observing three factors: long-run profit, stability of the workforce, and the level of capital investment in the new equipment. The goals in quantitative terms are: profit should be at least €125M, current level of employment of 4000 workers should be maintained, and the capital investment should not exceed €55M. Since goals may not be achievable, management decides to include the following penalties for the deviations. Penalty of 5 units for each €1M for missing the profit level; 2 units per 100 employees for going over employment goal and

4 units for going under the same goal; 3 units per €1M for exceeding the capital investment goal.

It is assumed that the contribution of each new product to profit, employment and capital investment level is proportional to the rate of production (linearity assumption). The contributions per unit rate are the following:

	Contribution				Penalty
	P1	P2	P3	Goal	per unit
Long-run profit	12	9	15	at least 125 (in millions)	5
Employment level	5	3	4	exactly 40 (in hundreds)	2(+), 4(–)
Capital investment	5	7	8	at most 55 (in millions)	3

Set up a *goal programming* model for the problem. Hint: watch for the nature of the different goals ('at least', 'exactly', 'at most').

12. The Father & Son haulage company is planning an extension of its fleet. Three types of trucks are included in the plan with the following characteristics:

Type	Load capacity (tons)	Cost (€1000)
Light	5	18
Medium	10	34
Large	20	55

Market analysis shows it would be desirable to add 10 light, 12 medium and 8 large models. The total capacity expansion should be around 300 tons and the total cost is limited to €1,000,000.

Write a *goal programming* model for the above problem if

- the financial constraint cannot be exceeded,
- it is equally undesirable to underachieve the number of light and medium models and overachieve the number of large models,
- it is undesirable to overachieve or underachieve the 300 ton goal of capacity expansion, underachievement being twice as bad as overachievement,

Explain your work.

Chapter 2

Graphical solution of linear programming problems

2.1 Exercises

1. Find graphically the feasible region of the following linear programming problem.

$$\begin{array}{lrcrcr} \text{max} & x_1 & + & x_2 & & \\ \text{subject to} & -0.5x_1 & + & x_2 & \leq & 3 \\ & 2x_1 & + & x_2 & \leq & 8 \\ & & & \multicolumn{3}{l}{x_1, x_2 \geq 0.} \end{array}$$

   Can you visually identify the optimal solution of this problem?

2. An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and body assembly shop. If the paint shop were only painting trucks, then 40 per day could be painted. If the paint shop were only painting cars, then 60 per day could be painted. If the body shop were only producing cars, then it could process 50 per day. If the body shop were only producing trucks, then it could process 50 per day. Each truck contributes $300 to profit, and each car contributes $200 to profit. Use LP to determine a daily production schedule that will maximize the company's profits.

3. Solve the following integer programming problem graphically:

$$\max\ x_1 + x_2$$

   subject to

$$
\begin{array}{rcrcr}
-10x_1 & + & 4x_2 & \leq & -3.0 \\
2.5x_1 & + & x_2 & \leq & 6.75 \\
5x_1 & - & 2x_2 & \leq & 7.5 \\
2.5x_1 & + & x_2 & \geq & 3.75 \\
\multicolumn{5}{c}{0 \leq x_1, x_2 \leq 3 \text{ and integer.}}
\end{array}
$$

4. Solve the following integer programming problem graphically:

$$\max\ x_1 + x_2$$

subject to

$$
\begin{array}{rcrcr}
-10x_1 & + & 4x_2 & \leq & -3.0 \\
2.5x_1 & + & x_2 & \leq & 6.75 \\
5x_1 & - & 2x_2 & \leq & 7.5 \\
2.5x_1 & + & x_2 & \geq & 3.75 \\
\multicolumn{5}{c}{0 \leq x_1, x_2 \leq 3 \text{ and integer.}}
\end{array}
$$

Chapter 3

Primal simplex: Phase II and Phase I

3.1 Summary of theoretical background for this chapter

We assume the problem is in CF #1.

Once the incoming nonbasic variable has been chosen the outgoing basic variable is determined by using an appropriate ratio test (different in Phase II and Phase I). Phase II assumes that a feasible basis is available and tries to improve the value of the objective function until optimality or unboundedness is detected. In Phase I, however, the basic solution is infeasible and the goal is to make basis changes that reduce the sum of infeasibilities until it reaches 0 at which point the solution becomes feasible and Phase II can commence.

Let $\boldsymbol{\alpha} = \mathbf{B}^{-1}\mathbf{a}$ be the transformed column vector of the entering variable. The basic solution is denoted by $\mathbf{x}_B \equiv \boldsymbol{\beta} = [\beta_1, \dots, \beta_m]^T$, the upper bound of the i-th basic variable is $\sigma_i,\ i = 1, \dots, m$.

3.1.1 Optimality conditions

Assume problem is in CF #1 and a feasible basis $\mathcal{B}$ is known The basic solution is $\mathbf{x}_\mathcal{B} = \mathbf{B}^{-1}(\mathbf{b} - \mathbf{R}\mathbf{x}_\mathcal{R})$, where $\mathcal{R}$ denotes the nonbasic (remaining) part of $\mathbf{A}$. Substituting this $\mathbf{x}_\mathcal{B}$ into the partitioned form of the objective function, $z = \mathbf{c}_\mathcal{B}^T\mathbf{x}_\mathcal{B} + \mathbf{c}_\mathcal{R}^T\mathbf{x}_\mathcal{R}$, we obtain $z = \mathbf{c}_\mathcal{B}^T\mathbf{B}^{-1}(\mathbf{b} - \mathbf{R}\mathbf{x}_\mathcal{R}) + \mathbf{c}_\mathcal{R}^T\mathbf{x}_\mathcal{R}$. Rearranged,

$$z = \mathbf{c}_\mathcal{B}^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_\mathcal{R}^T - \mathbf{c}_\mathcal{B}^T\mathbf{B}^{-1}\mathbf{R})\mathbf{x}_\mathcal{R}. \qquad (*)$$

In (*) the multiplier of nonbasic variable x_k is called the **reduced cost** of x_k

$$d_k = c_k - \mathbf{c}_\mathcal{B}^T\mathbf{B}^{-1}\mathbf{a}_k.$$

(*) shows that if the displacement of x_k is denoted by t the objective function changes by td_k, $z(t) = z + td_k$, i.e., the rate of the change is d_k. Since only feasible values of x_k are of interest, $\ell_k \le x_k + t \le u_k$ must hold for the displacement.

Sufficient conditions of optimality. Basis $\mathcal{B}$ is optimal if for every $k \in \mathcal{R}$:

Type(x_k)	Value	d_k	Remark
0	$x_k = 0$	Immaterial	
1	$x_k = 0$	≥ 0	
1	$x_k = u_k$	≤ 0	$k \in \mathcal{U}$
2	$x_k = 0$	≥ 0	
3	$x_k = 0$	$= 0$	

$\mathcal{U}$ is the index set of nonbasic variables at upper bound: $\mathcal{U} = \{j : x_j \in \mathcal{R}, x_j = u_j\}$.

3.1.2 Ratio test in phase II

Two sets of basic positions are defined:

$$\begin{aligned}\mathcal{I}^+ &= \{i : \alpha_i > 0,\ \text{type}(\beta_i) \in \{0,1,2\}\},\\ \mathcal{I}^- &= \{i : \alpha_i < 0,\ \text{type}(\beta_i) \in \{0,1\}\}.\end{aligned}$$

If the feasible displacement of the incoming variable is positive the following minima must be determined:

$$\begin{aligned}\theta^+ &= \min_{i\in\mathcal{I}^+}\{t_i\} = \min_{i\in\mathcal{I}^+}\left\{\frac{\beta_i}{\alpha_i}\right\}.\\ \theta^- &= \min_{i\in\mathcal{I}^-}\{t_i\} = \min_{i\in\mathcal{I}^-}\left\{\frac{\beta_i - \sigma_i}{\alpha_i}\right\}.\end{aligned}$$

Let u denote the upper bound of the incoming variable (can be finite or infinite). Determine

$$\theta = \min\{\theta^+, \theta^-, u\}.$$

If this minimum is achieved by θ^+ or θ^- then the basic variable that defined the minimum ratio leaves the basis (at lower or upper bound) and the entering variable takes its place in the basis. If the minimum is defined by u there is no basis change and the "incoming" variable does not become basic but goes to its other bound.

If the feasible displacement of the incoming variable is negative the above rules apply with α_i replaced by $-\alpha_i$.

3.1.3 Ratio test in phase I

Ratio test in phase I aims at improving the overall infeasibility of the solution, i.e., tries to reduce the sum of infeasibilities. This procedure requires an improving (incoming) candidate column (variable). Here we assume it is column $q \in \mathcal{R}$. Its updated form is $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$. The purpose of the ratio test is to determine the outgoing basic variable.

There are two main versions of the phase I ratio test. The traditional one determines the leaving variable such that the all basic variables at feasible level remain feasible. In practical terms, it takes the basic position that defines the smallest ratio.

The advanced one attempts to make the largest possible improvement towards feasibility that can be achieved with the chosen improving variable. It allows free (maybe infeasible) movements of basic variables. It may surpass the smallest ratio and consecutive ratios as long as the sum of infeasibilities keeps improving along the piece-wise linear function defined by the computed ratios (that act as break points).

Let $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ denote the basic positions of type(0), ..., type(3) basic variables and $\mathcal{I} = \bigcup_{i=0}^{3} \mathcal{I}_i$. Furthermore, $\mathcal{I}_\ell = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{I}_u = \mathcal{I}_0 \cup \mathcal{I}_1$. Obviously, $|\mathcal{I}| = m$. If the displacement of the incoming variable is positive the ratio test defines the following ratios (break points).

Basic variables reach their lower bound or go below it at

$$\tau_{\ell_i} = \begin{cases} \beta_i/\alpha_i > 0 & \text{if } \alpha_i \neq 0 \text{ and } i \in \mathcal{I}_\ell, \\ 0 & \text{if } \beta_i = 0,\ \alpha_i > 0 \text{ and } i \in \mathcal{I}_{\boldsymbol{\ell}}. \end{cases}$$

Basic variables reach their upper bound or go beyond it at

$$\tau_{u_i} = \begin{cases} (\beta_i - \sigma_i)/\alpha_i > 0 & \text{if } \alpha_i \neq 0 \text{ and } i \in \mathcal{I}_u, \\ 0 & \text{if } \beta_i = \sigma_i,\ \alpha_i < 0 \text{ and } i \in \mathcal{I}_u. \end{cases}$$

If $\sigma_i = 0$ then $\tau_{\ell_i} = \tau_{u_i}$ provided $\alpha_i \neq 0$.

Having determined the τ break points they have to be put in an ascending order:

$$0 \leq t_1 \leq \cdots \leq t_S,$$

where S denotes the number of break points and t is used with simple subscript to denote the elements of the ordered set of the τ_{ℓ_i} and τ_{u_i} values.

The traditional method chooses the smallest ratio, i.e., t_1 and the basic variable that defined t_1 leaves the basis.

The advanced method works in the following way. Let $r_1 = -d_q$, where d_q is the phase I reduced cost of variable q and compute

$$r_{k+1} = r_k - |\alpha_{j_k}|, \quad k = 1, 2, \ldots \tag{3.1}$$

Where α_{j_k} is the α component of the basic position that defined the k-th break point. Stop at index s for which

$$r_s > 0 \quad \text{and} \quad r_{s+1} \leq 0 \tag{3.2}$$

hold. t_s is the displacement of the incoming variable and the basic variable that defined t_s leaves the basis.

If the displacement of the incoming variable is negative the above procedure is still applicable with all occurrences of α replaced by $-\alpha$.

3.2 Exercises

1. Determine the type of each variable in the following problem. Is the given solution feasible? Do the variables satisfy the optimality conditions? Also, identify if the solution is degenerate and say why.

   Problem: $\max\ \mathbf{c}^T\mathbf{x},\ \mathbf{A}\mathbf{x} = \mathbf{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	$-\infty$	0	0	0
u_j	1	$+\infty$	$+\infty$	0	10	$+\infty$
type(x_j)						
In the solution:						
B/N	B	B	B	N	N	N
Value	1	1	-1	0	10	0
d_j	0	0	0	-10	10	0
Opt. cond. Y/N						

2. Determine the type of each variable in the following problem. Is the given solution feasible? Do the variables satisfy the optimality conditions? Also, identify if the solution is degenerate and say why.

   Problem: $\min\ \mathbf{c}^T\mathbf{x},\ \mathbf{A}\mathbf{x} = \mathbf{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	0	$-\infty$	0	0
u_j	0	$+\infty$	$+\infty$	$+\infty$	10	$+\infty$
type(x_j)						
In the solution:						
B/N	B	B	B	N	N	N
Value	0	1	11	0	10	0
d_j	0	0	0	0	10	10
Opt. cond. Y/N						

3. Show that the reduced cost of every basic variable is zero for any feasible basis $\mathbf{B}$ of a general linear programming problem.

4. Assume, we have an LP problem: $\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 11$ and a type-2 incoming variable, $x_q = 0$ with $d_q = -3$, are given. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. Is the BFS degenerate?

Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_q^i	t_i
1	0	3	$+\infty$	1	
2	2	2	$+\infty$	-1	
3	3	1	4	1	
4	8	2	$+\infty$	2	
5	2	1	6	-2	
6	4	2	$+\infty$	1	

5. Assume, we have an LP problem: $\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 10$ is given. A type-1 variable is coming in from its upper bound of 3. Its reduced cost is -2. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. Is the BFS degenerate?

   Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_q^i	t_i
1	0	3	$+\infty$	-1	
2	2	2	$+\infty$	1	
3	2	1	4	0	
4	8	2	$+\infty$	2	
5	0	0	0	0	

6. Assume, we have an LP problem: $\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{Ax} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 3$ and a type-2 incoming variable, $x_q = 0$ with $d_q = -4$, are given. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. Is the BFS degenerate?

   Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_q^i	t_i
1	2	2	$+\infty$	2	
2	2	2	$+\infty$	-1	
3	3	1	4	-1	
4	8	2	$+\infty$	2	
5	2	1	6	-2	

7. Solve the following linear programming problem using the simplex method with all types of variables

$$
\begin{array}{rrcrcrcl}
\min\ z = & -4x_1 & - & 2x_2 & - & 12x_3 & & \\
\text{s.t.} & x_1 & + & 3x_2 & - & 2x_3 & \leq & 12 \\
 & -3x_1 & + & x_2 & + & 2x_3 & = & 0 \\
-2 \leq & 4x_1 & - & x_2 & + & x_3 & \leq & 6 \\
 & \multicolumn{7}{l}{x_1, x_2 \geq 0,\ \ 0 \leq x_3 \leq 1}
\end{array}
$$

8. Let $\boldsymbol{\beta} = \mathbf{x}_B$ be a given infeasible basic solution and and type-2 incoming variable with the corresponding $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ column. Compute the phase-one reduced cost of the incoming variable, the value of the phase-one objective function, determine the outgoing variable, the steplength, the new objective value and the updated x_B values

   (*a*) using the traditional method,

   (*b*) using the advanced method.

i	type(x_{Bi})	x_{Bi}	u_{Bi}	α_q^i
1	0	2	0	-1
2	1	5	1	1
3	2	-14	∞	-7
4	0	3	0	1
5	2	-4	∞	-2
6	3	-2	∞	-2

9. Let $\boldsymbol{\beta} = \mathbf{x}_B$ be a given infeasible basic solution and and type-1 incoming variable (its upper bound value is 6) with the corresponding $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ column. Compute the phase-one reduced cost of the incoming variable, the value of the phase-one objective function, determine the outgoing variable, the steplength, the new objective value and the updated x_B values

   (*a*) using the traditional method,

(*b*) using the advanced method.

i	type(x_{Bi})	x_{Bi}	u_{Bi}	α_q^i
1	1	-2	5	0
2	0	-2	0	-1
3	1	5	1	-1
4	1	3	1	1
5	1	-40	2	4
6	3	-4	∞	-2
7	2	-3	∞	3

10. Find a feasible solution (if any) for the following linear programming problem.

$$\begin{array}{lrcrcrcr} \text{s.t.} & x_1 & - & 3x_2 & + & 2x_3 & \leq & -2 \\ & 3x_1 & + & 2x_2 & - & 2x_3 & \geq & 6 \\ & x_1 & & & - & x_3 & = & 3 \\ & & & 2x_2 & - & x_3 & \leq & -1 \end{array}$$

$$x_1 \geq 0,\ x_2 \leq 0,\ 0 \leq x_3 \leq 1.$$

11. Solve the following linear programming problems using the two phase primal simplex method.

$$\begin{array}{llrcrcrcr} \max & z = & -x_1 & + & 2x_2 & - & x_3 & & \\ \text{s.t.} & & 2x_1 & - & 2x_2 & + & x_3 & = & 6 \\ & & 3x_1 & - & 5x_2 & + & 2x_3 & \leq & 15 \\ & & x_1 & + & x_2 & - & x_3 & \geq & -3 \\ & & -x_1 & + & 3x_2 & - & x_3 & \leq & -1 \end{array}$$

$$x_1 \geq 0,\ x_2 \text{ free},\ 0 \leq x_3 \leq 2.$$

12. Solve the following linear programming problems using the two phase primal simplex method.

$$\begin{array}{llrcrcr} \min & z = & -x_1 & + & x_2 & & \\ \text{s.t.} & & 2x_1 & + & x_2 & \geq & 1 \\ & & x_1 & + & x_2 & \leq & 3 \\ & & x_1 & - & x_2 & \geq & -1 \end{array}$$

$$x_1, x_2 \geq 0$$

Chapter 4

The dual simplex algorithm: Phase II and Phase I

4.1 Summary of theoretical background for this chapter

We assume the problem is in computational form (CF) #1.

Phase II of the dual simplex method moves on dual feasible bases until optimality or dual unboundedness (primal infeasibility) is detected. In dual phase I a search for a dual feasible basis is performed.

Dual algorithms choose the outgoing variable first such that the phase specific objective function (dual objective function or sum of dual infeasibilities) can improve, then use a phase specific ratio test to determine the incoming variable.

Dual feasibility conditions are identical to the primal optimality conditions (see chapter 3).

Assume the p-th basic variable (x_{Bp}) is chosen to leave the basis. The updated (transformed) pivot row is $\boldsymbol{\alpha}^p = \boldsymbol{\rho}_p^T \mathbf{R} = \mathbf{e}_p^T \mathbf{B}^{-1} \mathbf{R}$, where $\mathbf{R}$ is the nonbasic (remaining) part of $\mathbf{A}$.

In dual phase II indices participating in the ratio test are:

If $x_{Bp} < 0$

$$J = \{j \,|\, (\alpha_j^p < 0 \wedge x_j = 0) \vee (\alpha_j^p > 0 \wedge x_j = u_j)\}.$$

If $x_{Bp} > u_{Bp}$

$$J = \{j \,|\, (\alpha_j^p > 0 \wedge x_j = 0) \vee (\alpha_j^p < 0 \wedge x_j = u_j)\}.$$

The standard ratio test determines the dual steplength and the index q of the entering variable:

$$\theta_D = \left| \frac{d_q}{\alpha_q^p} \right| = \min_{j \in J} \left| \frac{d_j}{\alpha_j^p} \right|,$$

where d_j-s are the components of the updated objective row.

The main logic of the dual phase II algorithm determines the α_q^p pivot element. After that a Gauss-Jordan elimination is performed (including the objective row) using this pivot.

Remarks:

- Type-0 variables are not involved in the ratio test because any value of the corresponding d_j is dual feasible (DF).
- Type-3 variables are always involved in the dual ratio test if $\alpha_j^p \neq 0$ and they determine a 0 ratio.
- In this chapter we only deal with the standard ratio test (smallest ratio). While there exists a generalized version as well it is rather involved and lies beyond the scope of these notes.

Dual phase II algorithm is the main computational engine in (mixed) integer programming if branch and bound type procedure is used (which is nearly almost the case).

Dual phase I algorithm is relatively complicated. It has a standard and also an advanced version. They are not covered in the current version of the notes.

4.2 Exercises

1. Formulate the dual of the following problem.

$$
\begin{array}{rrrrrcr}
\max z = & x_1 - & 2x_2 + & 3x_3 - & 4x_4 & & \\
\text{s.t.} & 2x_1 + & 5x_2 - & 4x_3 + & 9x_4 & \leq & 15 \\
& x_1 + & 4x_2 + & 2x_3 - & 6x_4 & \geq & -7 \\
& 4x_1 - & 3x_2 - & 6x_3 + & 4x_4 & = & 1 \\
& & & x_1, \ldots, x_4 \geq 0. & & &
\end{array}
$$

2. Write the dual of the problem given by

$$
\begin{array}{rrrrcr}
\min z = & -x_1 + & 2x_2 + & 6x_3 & & \\
\text{s.t.} & x_1 + & 3x_2 - & 2x_3 & \geq & 0 \\
& -x_1 - & 2x_2 + & 5x_3 & = & 0 \\
& 2x_1 + & 3x_2 + & 4x_3 & \leq & 0 \\
& & x_1 \leq 0, & x_2, x_3 \geq 0. & &
\end{array}
$$

3. Formulate the dual of the following linear programming problem

$$
\begin{array}{lll}
(\text{P}) & \min & \mathbf{c}^T\mathbf{x} \\
& \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \\
& & \mathbf{x} \geq \mathbf{0}.
\end{array}
$$

   Show that the dual of the dual is the original primal problem (P).

4. The weak duality theorem says that if an arbitrary primal feasible solution $\mathbf{x}$ is given then for any dual feasible solution $\mathbf{y}$ the relation $\mathbf{b}^T\mathbf{y} \leq \mathbf{c}^T\mathbf{x}$ holds. The strong duality theorem says that if a problem has a feasible finite solution then its dual pair has a feasible finite optimum too, and the objective values are the same. Prove these theorems.

5. Show that if the primal has an unbounded solution the dual problem has no feasible solution (the dual problem is infeasible). Use the following primal-dual pair:

$$\begin{array}{rr} (P) & \min\ \mathbf{c}^T\mathbf{x} \\ s.t. & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad\qquad \begin{array}{rr} (D) & \max\ \mathbf{b}^T\mathbf{y} \\ s.t. & \mathbf{A}^T\mathbf{y} \leq \mathbf{c} \end{array}$$

6. Investigate whether the following linear programming problem can be solved with phase II of the dual simplex method. Explain your answer. If yes, convert the problem into a form needed by the algorithm and solve it. Discuss the solution steps in detail. Provide solutions for both the primal and dual.

$$\begin{array}{lrcrcrcrcr} \text{max} & x_1 & - & 4x_2 & - & 2x_3 & - & 2x_4 & & \\ \text{s. t.} & -2x_1 & - & x_2 & - & x_3 & & & \leq & -1 \\ & 2x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 3 \\ & & & 4x_2 & - & x_3 & - & 2x_4 & \leq & -2 \\ & \multicolumn{9}{c}{x_1 \leq 0,\ x_j \geq 0,\ j = 2, \ldots, 4} \end{array}$$

7. Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{lrcrcrcrcr} \text{min} & x_1 & - & 2x_2 & + & 4x_3 & + & 4x_4 & & \\ \text{s. t.} & 2x_1 & + & 4x_2 & - & 4x_3 & & & \leq & -1 \\ & x_1 & + & 4x_2 & & & + & 2x_4 & \geq & 2 \\ & x_1 & - & x_2 & + & x_3 & + & x_4 & \leq & 3 \\ & \multicolumn{9}{c}{x_j \geq 0,\ j = 1, 3, 4,\ x_2 \leq 0.} \end{array}$$

8. Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{llrcrcrcrcr} \text{min} & z = & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & & \\ \text{s.t.} & & x_1 & & & & & + & x_4 & \geq & 4 \\ & & x_1 & + & x_2 & & & & & \geq & 8 \\ & & & & x_2 & + & x_3 & & & \geq & 8 \\ & & & & & & x_3 & + & x_4 & \geq & 6 \end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

9. Solve the following primal-dual pair with using the primal and the dual simplex algorithms. Verify that the strong duality theorem holds for the solutions:

$$\begin{array}{llrcrcrcr}
\max & z = & & & -x_2 & - & 2x_3 & & \\
\text{s.t.} & & x_1 & + & x_2 & - & 2x_3 & \leq & -2 \\
& & 2x_1 & - & x_2 & + & x_3 & \leq & -4
\end{array}$$
$$x_1, x_2, x_3 \geq 0.$$

$$\begin{array}{llrcrcr}
\min & w = & -2y_1 & - & 4y_2 & & \\
\text{s.t.} & & y_1 & + & 2y_2 & \geq & 0 \\
& & y_1 & - & y_2 & \geq & -1 \\
& & -2y_1 & + & y_2 & \geq & -2
\end{array}$$
$$y_1, y_2 \geq 0.$$

10. Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{llrcrcrcrcr}
\max & z = & -2x_1 & - & x_2 & - & 3x_3 & - & x_4 & & \\
\text{s.t.} & & -x_1 & + & 2x_2 & + & x_3 & - & x_4 & \leq & 3 \\
& & -x_1 & - & x_2 & + & x_3 & + & 2x_4 & \geq & 2 \\
& & x_1 & - & 2x_2 & - & 3x_3 & - & x_4 & \leq & -2
\end{array}$$
$$x_1, \ldots, x_4 \geq 0.$$

Chapter 5

Integer and mixed integer linear programming

5.1 Exercises

1. In some model we have a variable x that is allowed to take a value from the following set $F = \{0, -1, -2, -3\}$. How can you formulate this requirement with integer programming constraint(s) and/or bounds?

2. Convert the following discrete optimization problem into a mixed integer linear programming problem.

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & y_1 + y_2 + \ldots + y_r = 1 \\ & \mathbf{x} \geq \mathbf{0} \\ & x_1 \in \{r_1, r_2, \ldots, r_q\}. \end{array}$$

   Note: $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$.

3. A trading company is considering four investments: Investment 1 will yield a net present value (NPV) of \$16,000; investment 2, an NPV of \$22,000; investment 3, an NPV of \$12,000; and investment 4, an NPV of \$8,000. Each investment requires a certain cash outflow at the present time: investment 1, \$5,000; investment 2, \$7,000; investment 3, \$4,000; and investment 4, \$3,000. Currently, \$14,000 is available for investment.

   (a) Formulate an IP whose solution will tell the company how to maximize the NPV obtained from investments 1–4.

   (b) Modify the formulation to account for each of the following requirements:

(*i*) The company can invest in at most two investments.
(*ii*) Investment 2 can be carried out only if investment 1 is done.
(*iii*) If investment 2 is selected, they can't invest in investment 4.

4. The following euro coins are available: $1, 2, 5, 10, 20, 50$ cents and $1, 2$ euros. Write a mathematical model to find the minimum number of coins needed to pay a given quantity q expressed in euros.

5. A furniture company is capable of manufacturing three types of furniture: chair, desk, and cabinets. The manufacturing of each type of furniture requires to have the appropriate type of production line available. The line needed to manufacture each type of furniture must be rented at the following rates: chair line, \$200 per week; desk line, \$150 per week; cabinet line, \$100 per week. The chair line can produce a maximum of 40 chairs per week, the desk line can produce a maximum of 53 desks per week, and the cabinet line can produce a maximum of 25 cabinets per week. The manufacture of each type of furniture also requires some amount of wood and labor as shown below. Each week, 150 hours of labor and 160 m^2 of wood are available. The variable unit cost and selling price for each type of furniture are also give. Formulate an IP whose solution will maximize the company's weekly profits.

Furniture Type	Labor (Hours)	Wood (m^2)	Sales Price (\$)	Variable Cost (\$)
Chair	2	3	8	4
Desk	3	4	12	6
Cabinet	6	4	15	8

6. You have 5 keys and 6 locks. Every key opens one or more locks as shown in the following table:

	Key1	Key2	Key3	Key4	Key5
Lock1	x	x	x		
Lock2			x	x	
Lock3	x				x
Lock4	x	x			x
Lock5		x		x	
Lock6		x		x	x

Write an optimization model that chooses the minimum number of keys such that any of the locks can be opened.

7. Company wants to build plants to supply customers. There are m customers and n potential locations for plants.

   Problem data:

n	potential locations for plants
m	number of customers
c_{ij}	cost of supplying one unit of demand i from plant j
f_j	fixed cost of opening (building) a plant in location j
d_i	demand of customer i
s_j	supply available at plant j (if open)

   Decision variables:

x_j^i	units of product delivered from plant j to customer i
y_j	binary variable: $= 1$ if plant j is to be built, 0 otherwise.

   Formulate a mixed integer LP problem to minimize costs.

8. A car manufacturing company is considering the production of three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by each type of car are given in the table. Currently, 6,000 tons of steel and 60,000 hours of labor are available. For production of a type of a car to be economically feasible, at least 1,000 cars of that type must be produced. Formulate an IP model to maximize the company's profit.

	Car Type		
Resource	Compact	Midsize	Large
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit ($)	2,000	3,000	4,000

9. Solve the following two dimensional mixed integer linear programming problem graphically using your own drawing in a graph similar to the one below. It need not be very accurate. If in doubt, rely on the given numerical data.

   The objective is to maximize $z = -x_1 + 2x_2$, where x_1 is a general nonnegative integer, x_2 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $P_1(0,0)$, $P_2(0,1)$, $P_3(1,3)$, $P_4(3,4)$, $P_5(4,3)$ and $P_6(2,0)$. Where are the feasible solutions of the problem located? Determine an optimal solution. Is it unique? If not, can you find them all? How many are there? Compare the situation with continuous LP.

10. Find graphically the feasible region of the following integer linear programming problem.

$$\begin{array}{lrcrcr} \text{min} & -3x & - & 4y & & \\ \text{s.t.} & x & + & 2y & \leq & 10 \\ & x & + & y & \leq & 7 \\ & \multicolumn{5}{c}{0 \leq x \leq 6,\ 0 \leq y \leq 4} \\ & \multicolumn{5}{c}{x, y \text{ integer}} \end{array}$$

(*i*) Can you visually identify the optimal solution of this problem?

(*ii*) What is the optimal solution if the $2x + 2y \leq 9$ additional constraint is also imposed on the LP?

11. Solve the following integer programming problem graphically:

$$\begin{array}{lrcrcr} \text{max} & x & + & y & & \\ \text{s.t.} & -10x & + & 4y & \leq & -3.0 \\ & 2.5x & + & y & \leq & 6.75 \\ & 5x & - & 2y & \leq & 7.5 \\ & 2.5x & + & y & \geq & 3.75 \\ & \multicolumn{5}{c}{0 \leq x, y \leq 3 \text{ and integer.}} \end{array}$$

12. The objective is to maximize $z = 2x_1 + x_2$, where x_2 is a general nonnegative integer, x_1 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $P_1(0, 1),\ P_2(0, 3.5)$ and $P_3(2.95, 0)$. Where are the feasible solutions located? Determine an optimal solution graphically. Is it unique?

Chapter 6

Branch-and-bound techniques, cutting plane algorithms

6.1 Exercises

1. Solve the following integer programming problem graphically using the B&B method.

$$\begin{array}{rrcrcl}
\text{min } z = & -3x_1 & - & 4x_2 & + & 20 \\
\text{s.t.} & \frac{2}{5}x_1 & + & x_2 & \leq & 3 \\
& \frac{2}{5}x_1 & - & \frac{2}{5}x_2 & \leq & 1 \\
\end{array}$$
$$x_1, x_2 \geq 0 \text{ and integer.}$$

2. Use branch-and-bound algorithm (B&B) to solve the following IP:

$$\begin{array}{llrcrcrcrcl}
\text{min} & z = & -3x_1 & - & x_2 & - & x_3 & - & x_4 & & \\
\text{s.t.} & & 5x_1 & + & 2x_2 & + & 2x_3 & + & 2x_4 & \leq & 5
\end{array}$$
$$x_1, x_2, x_3, x_4 \in \{0, 1\}$$

3. Use branch-and-bound algorithm (B&B) to solve the following IP graphically:

$$\begin{array}{llrcrcl}
\text{max} & z = & 8x_1 & + & 5x_2 & & \\
\text{s.t.} & & x_1 & + & x_2 & \leq & 6 \\
& & 9x_1 & + & 5x_2 & \leq & 45
\end{array}$$
$$x_1, x_2 \geq 0;\ x_1, x_2 \text{ integer}$$

4. Use branch-and-bound (B&B) algorithm and Simplex Methods to solve the following IP:

$$\begin{array}{llrcrcr}
\text{max} & z = & x_1 & + & 4x_2 & & \\
\text{s.t.} & & 5x_1 & + & 8x_2 & \leq & 40 \\
 & & -2x_1 & + & 3x_2 & \leq & 9
\end{array}$$

$$x_1, x_2 \geq 0;\ x_1, x_2 \text{ integer}$$

Chapter 7

Network optimization

7.1 Exercises

1. Give the mathematical model of the transportation problem.

2. Find a starting basis of the following transportation problem using the North-West Corner Rule Method. Is this basis optimal or not? Why?

$$\mathbf{s} = \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}, \qquad \mathbf{d} = (3, 5, 4, 1), \qquad \mathbf{C} = \begin{pmatrix} 4 & 2 & 5 & 6 \\ 4 & 1 & 3 & 7 \\ 8 & 6 & 5 & 4 \end{pmatrix}$$

3.

4. Find a starting basis of the following transportation problem using the Least Cost Cell Method. Is this basis optimal or not? Why?

$$\mathbf{s} = \begin{pmatrix} 20 \\ 12 \\ 30 \end{pmatrix}, \qquad \mathbf{d} = (15, 20, 15, 12), \qquad \mathbf{C} = \begin{pmatrix} 9 & 7 & 6 & 6 \\ 8 & 6 & 7 & 9 \\ 7 & 8 & 8 & 5 \end{pmatrix}$$

5. Solve the following transportation problem using the North-West Corner Rule Method.

$$\mathbf{s} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \qquad \mathbf{d} = (2, 4, 2, 2, 2), \qquad \mathbf{C} = \begin{pmatrix} 2 & 3 & 4 & 1 & 2 \\ 4 & 5 & 3 & 2 & 1 \\ 1 & 3 & 4 & 6 & 2 \end{pmatrix}$$

6. Solve the following transportation problem using the Least Cost Cell Method.

$$\mathbf{s} = \begin{pmatrix} 10 \\ 7 \\ 12 \\ 11 \end{pmatrix}, \qquad \mathbf{d} = (10, 10, 10, 10, 10, 10), \qquad \mathbf{C} = \begin{pmatrix} 2 & 6 & 5 & 3 \\ 4 & 3 & 4 & 2 \\ 2 & 6 & 4 & 4 \\ 6 & 8 & 7 & 9 \end{pmatrix}$$

7. Solve the following transportation problem using the North-West Corner Rule Method.

$$\mathbf{s} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix}, \qquad \mathbf{d} = (4, 4, 4, 4, 4, 4), \qquad \mathbf{C} = \begin{pmatrix} 4 & 3 & 2 & 5 & 7 & 2 \\ 3 & 4 & 3 & 5 & 3 & 7 \\ 6 & 5 & 4 & 3 & 4 & 2 \end{pmatrix}$$

8. Solve the following transportation problem using the Least Cost Cell Method.

$$\mathbf{s} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix}, \qquad \mathbf{d} = (4, 4, 4, 4, 4, 4), \qquad \mathbf{C} = \begin{pmatrix} 4 & 3 & 2 & 5 & 7 & 2 \\ 3 & 4 & 3 & 5 & 3 & 7 \\ 6 & 5 & 4 & 3 & 4 & 2 \end{pmatrix}$$

9. Solve the following transportation problem, where $x_{11} = x_{12} = x_{23} = x_{34} = 0$ (x_{ij} is the quantity transported from node i to node j).

$$\mathbf{s} = \begin{pmatrix} 7 \\ 6 \\ 8 \end{pmatrix}, \qquad \mathbf{d} = (5, 3, 5, 5, 3), \qquad \mathbf{C} = \begin{pmatrix} 2 & 3 & 4 & 2 & 5 \\ 3 & 3 & 1 & 4 & 3 \\ 2 & 2 & 4 & 3 & 4 \end{pmatrix}$$

10. Suppose that a taxi firm has four taxis available, and four customers wishing to be picked up as soon as possible. The firm prides itself on speedy pickups, so for each taxi the "cost" of picking up a particular customer will depend on the time taken for the taxi to reach the pickup point (see the "cost" matrix $\mathbf{C}$, where c_{ij} defines the distance in time between the taxi i and customer j). Give an optimal "taxi-customer" assignment where the total waiting time of the customers is minimal.

$$\mathbf{C} = \begin{pmatrix} 14 & 5 & 8 & 7 \\ 2 & 12 & 6 & 5 \\ 7 & 8 & 3 & 9 \\ 2 & 4 & 6 & 10 \end{pmatrix}$$

Chapter 8

Game theory

8.1 Exercises

1. A 2p0sg has the following reward matrix:

	C's strategy		
R's strategy	C1	C2	C3
R1	17	23	48
R2	17	3	51
R3	6	17	3

   Which strategy should each of the two players choose? One answer must be obtained by applying the concept of dominated strategies to rule out a succession of inferior strategies until only one choice remains.

2. Three linear functions $y_1,\ y_2$ and y_3 are defined as follows:

$$\begin{aligned} y_1 &= 2 - x \\ y_2 &= x - 1 \\ y_3 &= 2x - 6 \end{aligned}$$

   Find $\min\limits_{x\geq 0} \max\limits_{i} \{y_i\}$.

3. The manager of a multinational company and the union of workers are preparing to sit down at the bargaining table to work out the details of a new contract for the workers. Each side has developed certain proposals for the contents of the new contract. Let us call union proposals "Prop-1", "Prop-2" and "Prop-3, and the manager proposals "Contr-A" (for contract), "Contr-B" and "Contr-C". Both parties are aware of the financial consequences of each proposalontract combination. The pay-off matrix is:

	Manager's		
Workers'	Contr-A	Contr-B	Contr-C
Prop-1	8.5	7.0	7.5
Prop-2	12.0	9.5	9.0
Prop-3	9.0	11.0	8.0

These values are the contract gains that the workers' union would secure and also the cost the company would have to bear.

Is there a clearut contract combination agreeable to both parties, or will they find it necessary to submit to arbitration in order to arrive at some sort of compromise?

4. Consider the same situation as in Problem 3, but with the following pay-off matrix:

	Manager's		
Workers'	Contr-A	Contr-B	Contr-C
Prop-1	9.5	12.0	7.0
Prop-2	7.0	8.5	6.5
Prop-3	6.0	9.0	10.0

Is there an equilibrium point?

Find the mixed strategies for the union and the manager.

Formulate (but do not solve) the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.

Chapter 9

Nonlinear programming

1. Determine whether the following functions are convex or not for $x \in \mathbb{R}^1$:

$$f(x) = 1 + 2x + x^2, \qquad g(x) = x^2 + e^{-x}, \qquad h(x) = x^2 - e^x.$$

2. Determine whether the following functions are convex for $\mathbf{x} > \mathbf{0}$. Note, $\mathbf{x} = (x_1, x_2)$.

$$f(x_1, x_2) = 4x_1^2 - 4x_1x_2 + x_2^2 - \log(x_1), \quad g(x_1, x_2) = 4x_1^2 + x_2^2 + 4x_1x_2 + \log(x_1x_2).$$

3. Show that the following function is convex and determine its minimum

$$f(x) = \frac{11}{273}x^6 - \frac{19}{91}x^4 + x^2.$$

4. A furniture company makes wall cabinets. There is a fixed cost of production per month of €6000. The cost of making a chair is €30. Sales price affects the quantities sold:

$$\text{volume}(v) = 500 - 1.4\,\text{price}(p).$$

Work out a profit function and determine the price that will maximize profit. Also, compute the optimum value.

5. Find the extreme points of $f(x) = x^4 - 2x^2 + 2$. Determine whether they are local or global minima/maxima. Having done so, determine the minimum of the same function $f(x)$ subject to $-0.5 \leq x \leq 1.5$.

6. Find the minimum of $g(x) = x^2 + e^{-x}$.

7. Solve the following nonlinear programming problem.

$$\min\ f(x) = \frac{1}{4}x^2 + x + 1, \quad \text{subject to } -1 \leq x \leq 2.$$

8. Determine which of the following functions is smooth/nonsmooth on the given domain. [Note: $f(\mathbf{x}) = f(x_1, x_2)$.]

   (i) $f(\mathbf{x}) = \log(x_1 x_2) - (x_1 + x_2)^2, \quad 0 < x_1, x_2 \leq 100$

   (ii) $g(\mathbf{x}) = |x_1 - 2| + x_2^3, \quad 0 \leq x_1, x_2 \leq +\infty$

   (iii) $h(\mathbf{x}) = |x_1 + x_2|^2, \quad -\infty \leq x_1, x_2 \leq +\infty$

9. Which of the following functions have local extreme points (minimum or maximum), and if so, where? Why? [Note: $f(\mathbf{x}) = f(x_1, x_2)$.]

   (i) $f(\mathbf{x}) = 1 - x_1 x_2$

   (ii) $f(\mathbf{x}) = x_1^2 - x_2^3$

   (iii) $f(\mathbf{x}) = x_1^2 + x_2^2$

10. Write the KKT conditions for the following problem:

$$
\begin{aligned}
\min\ f(\mathbf{x}) \quad &= \quad x_1^4 + 2x_1^2 + 2x_1 x_2 + 4x_2^2 \\
s.t. \quad & \quad 2x_1 + x_2 = 10 \\
& \quad x_1 + 2x_2 \geq 10 \\
& \quad x_1, x_2 \geq 0.
\end{aligned}
$$

11. Consider the following constrained nonlinear programming problem:

$$
\begin{aligned}
\max\ f(\mathbf{x}) \quad &= \quad x_1 x_2 + x_1 x_3 + x_2 x_3 \\
s.t. \quad & \quad x_1 + x_2 + x_3 = 3.
\end{aligned}
$$

    Define the KKT conditions for the problem. Find a solution that satisfies the conditions. Determine if it is a maximizer.

12. Consider the following nonlinear programming problem:

$$
\begin{aligned}
\min \quad & 4(x_1 - 2)^2 + (x_2 - 1)^2 \\
s.t. \quad & 16x_1 + 6x_2 = 63.
\end{aligned}
$$

    Write the $\mathcal{L}$ Lagrangian function of the problem. Define the necessary condition of optimality for $\mathcal{L}$ and solve the resulting system.

Part II

Solutions

Chapter 1

Basics of background of operations research

1.1 Linear algebra

1. What does triangle inequality say for the norms of two m dimensional vectors $\mathbf{a}$ and $\mathbf{b}$?

   Answer: $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$ for any vector norm.

2. Which of the following pairs of vectors are orthogonal? Why?

   (*a*) $[1, 2]$ and $[-1, 1]$,

   (*b*) $[2, 5, 1]$ and $[-3, 1, 1]$,

   (*c*) $[0, 1, -1.98]$ and $[1, 0.99, 1/2]$,

   (*d*) $[3, 5, 3, -4]$ and $[4, -2, 2, 2]$.

   Answer: For orthogonality of $\mathbf{u}$ and $\mathbf{v}$ the dot product must be zero: $\mathbf{u}^T\mathbf{v} = 0$. Therefore, (*a*) not orthogonal, (*b*), (*c*) and (*d*) are orthogonal.

3. Express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

   (*a*) $\mathbf{b} = [4, 5]$, $\mathbf{a}_1 = [1, 3]^T$ and $\mathbf{a}_2 = [2, 2]^T$,

   (*b*) $\mathbf{b} = [1, -2]$, $\mathbf{a}_1 = [2, 1]^T$ and $\mathbf{a}_2 = [5, 5]^T$,

   (*c*) $\mathbf{b} = [1, -2]$, $\mathbf{a}_1 = [2, -3]^T$ and $\mathbf{a}_2 = [2, -8]^T$,

   (*d*) $\mathbf{b} = [2, -15]$, $\mathbf{a}_1 = [3, -4]^T$ and $\mathbf{a}_2 = [14, 6]^T$.

   Answer: (*a*) $1/2\mathbf{a}_1 + 7/4\mathbf{a}_2$, (*b*) $3\mathbf{a}_1 - \mathbf{a}_2$., (*c*) $4/10\mathbf{a}_+1/10\mathbf{a}_2$, (*d*) $3\mathbf{a}_1 - 1/2\mathbf{a}_2$.

4. Which of the following sets of vectors are linearly independent:

(i) $[1, 5]$, $[2, 3]$;

(ii) $[2, 1, -3]$, $[-1, 1, -6]$, $[1, 1, -4]$.

Answer: Computing the determinants of the two sets, for *(i)* it is $\neq 0$, thus $[1, 5]$ and $[2, 3]$ are linearly independent, for the *(ii)* it is $= 0$, thus $[2, 1, -3]$, $[-1, 1, -6]$ and $[1, 1, -4]$ are linearly dependent.

5. Show that vectors $\mathbf{a}_1 = [2, 3, 1]^T,\ \mathbf{a}_2 = [1, 0, 4]^T,\ \mathbf{a}_3 = [2, 4, 1]^T,\ \mathbf{a}_4 = [0, 3, 2]^T$ are linearly dependent.

   Answer: No computations are needed. Here are 4 vectors all in $\mathbb{R}^3$. Only 3 of them can be linearly independent. The fourth will be linearly dependent.

6. For each of the following statements, determine whether it is true or false. Justify your answer.

   (*i*) A basis must contain $\mathbf{0}$.

   (*ii*) Subsets of linearly dependent sets are linearly dependent.

   (*iii*) Subsets of linearly independent sets are linearly independent.

   (*iv*) If $\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$ then all scalars λ_j are zero.

   (*v*) Any set of m vectors containing the null vector is linearly dependent.

   (*vi*) The dot product of two, linearly dependent, nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ is always equal to zero.

   (*vii*) If a matrix is multiplied by a diagonal matrix, the result does not depend on the order of multiplication.

   (*viii*) The product of two square matrices is always defined.

   Answer: (*i*) false, (*ii*) false, (*iii*) true, (*iv*) false, (*v*) true, as 0 can always be added with nonzero multiplier, (*vi*) false, as two linearly dependent vectors are $\mathbf{a}$ and $\lambda\mathbf{a},\ \lambda \neq 0$ and $\mathbf{a}^T\lambda\mathbf{a} = \lambda\mathbf{a}^T\mathbf{a} \neq 0$ if $\mathbf{a} \neq 0$, (*vii*) false, (*viii*) false, as they can be of different dimensions.

7. Answer the following questions:

   (*a*) How is the rank of an $m \times n$ matrix defined?

   (*b*) What is the relationship between the *row rank* and *column rank* of an $m \times n$ matrix $\mathbf{A}$?

   (*c*) What does *full rank* of an $m \times n$ matrix $\mathbf{A}$ mean?

   Answer: (*a*) The column rank of a matrix $\mathbf{A}$ is the maximal number of linearly independent columns of $\mathbf{A}$, the row rank is the maximal number of linearly independent rows of $\mathbf{A}$. As they are always equal, they are simply called the rank ρ of $\mathbf{A}$. It follows that $\rho(\mathbf{A}) \leq \min\{m, n\}$., (*b*) They are equal and they are called the rank of the matrix., (*c*) $\rho(\mathbf{A}) = \min\{m, n\}$.

8. Answer the following questions:

   (*a*) How is the basis of a vector space defined?

   (*b*) What is the size of a basis in $\mathbb{R}^n$? Is a basis unique for $\mathbb{R}^n$?

   (*c*) Which, if any, of the following systems of vectors are bases in $\mathbb{R}^3$:

   (i) $[1, 3, 2], [3, 1, 3], [2, 10, 2]$.

   (ii) $[1, 2, 1], [1, 0, 2], [2, 1, 1]$.

   Answer: (*a*) A basis is a set of vectors that, in a linear combination, can represent every vector in a given vector space (any vector in the space can be expressed as a linear combination of the vectors in the basis.)., (*b*) Any set of n linearly independent n dimensional vectors is a basis for $\mathbb{R}^n$. Thus, there can be infinitely many bases., (*c*) To form a basis the vectors must be linearly independent. Determinant of set in *(i)* is zero, the vectors are linearly dependent, no basis, determinant of set in *(ii)* is nonzero, vectors form a basis.

9. Answer the following questions:

   (*a*) How is the p-norm of a vector $\mathbf{v} \in \mathbb{R}^m$ defined? What are the important special cases?

   (*b*) What does the triangle inequality say for the norms of two compatible matrices $\mathbf{A}$ and $\mathbf{B}$?

   Answer: (*a*) $\|\mathbf{x}\|_p = \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p}$. Important special cases are $p = 1, 2, \infty$, giving $\|\mathbf{x}\|_1 = \sum_{i=1}^{m} |x_i|$, $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{m} x_i^2}$, and $\|\mathbf{x}\|_\infty = \max_{1 \le i \le m} |x_i|$., (*b*) $\|\mathbf{A} + \mathbf{B}\| \le \|\mathbf{A}\| + \|\mathbf{B}\|$ for any matrix norm.

10. Determine norms $\|\mathbf{A}\|_1$ and $\|\mathbf{A}\|_\infty$ of the given matrix $\mathbf{A}$:

    (*a*)

$$\mathbf{A} = \begin{bmatrix} -9 & 2 & 3 \\ -4 & 8 & 6 \\ 1 & 5 & 7 \end{bmatrix}.$$

    (*b*)

$$\mathbf{A} = \begin{bmatrix} 5 & -8 & 3 \\ -10 & 2 & 1 \\ 1 & 6 & -8 \end{bmatrix}.$$

(*c*)

$$\mathbf{A} = \begin{bmatrix} 3 & 4 & -2 \\ -6 & -4 & 1 \\ 4 & 3 & 9 \end{bmatrix}.$$

(*d*)

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & -2 \\ -6 & 4 & 1 \\ 3 & -3 & 7 \end{bmatrix}.$$

Answer: As $\|\mathbf{A}\|_1 = \max_j \|\mathbf{a}_j\|_1$ is the maximum absolute column sum and $\|\mathbf{A}\|_\infty = \max_i \|\mathbf{a}^i\|_1$ is the maximum absolute row sum: (*a*) $\|\mathbf{A}\|_1 = 16$ and $\|\mathbf{A}\|_\infty = 18$, (*b*) $\|\mathbf{A}\|_1 = 16$ and $\|\mathbf{A}\|_\infty = 16$ (accidental coincidence), (*c*) $\|\mathbf{A}\|_1 = 13$ and $\|\mathbf{A}\|_\infty = 16$, (*d*) $\|\mathbf{A}\|_1 = 12$ and $\|\mathbf{A}\|_\infty = 13$.

11. Solve the following system of equations using Gauss-Jordan elimination. Identify basic variables. Express all solutions in terms of non-basic variables.

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & - & x_5 & = & -2 \\ 4x_1 & + & 2x_2 & & & + & 3x_4 & - & 2x_5 & = & 2 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 3 \end{array}$$

Answer: After pivoting down the diagonal:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3/2 & 3 & -1 \\ 0 & 0 & 1 & -1/2 & 0 & 3 \end{array}\right],$$

Basic variables are x_1, x_2 and x_3. They are expressed in terms of nonbasic variables as: $x_1 = 1 + 2x_5,\ x_2 = -1 - \frac{3}{2}x_4 - 3x_5,\ x_3 = 3 + \frac{1}{2}x_4$, with arbitrary values for x_4 and x_5.

12. For

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

decide which of the following products are defined, and compute them:

(*a*) $\mathbf{Au}$, (*b*) $\mathbf{Av}$, (*c*) $\mathbf{A}^T\mathbf{v}$, (*d*) $\mathbf{u}^T\mathbf{v}$, (*e*) $\mathbf{uv}^T$.

Answer: (*a*) $\mathbf{Au} = [-3, -4]^T$, (*b*) $\mathbf{Av}$ not defined, (*c*) $\mathbf{A}^T\mathbf{v} = [-7, 6, 23]^T$, (*d*) $\mathbf{u}^T\mathbf{v}$ not defined as $\mathbf{u} \in \mathbb{R}^3,\ \mathbf{v} \in \mathbb{R}^2$, (e) $\mathbf{uv}^T = \begin{bmatrix} 2 & 3 \\ 4 & 6 \\ -2 & -3 \end{bmatrix}$, which is the outer product of $\mathbf{u}$ and $\mathbf{v}$.

13. Given matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix},$$

where $a,\ b,\ c$ and d are scalars. Compute $\mathbf{AB} - \mathbf{BA}$.

Give conditions for $\mathbf{AB} = \mathbf{BA}$.

Answer: $\mathbf{AB} - \mathbf{BA} = \begin{bmatrix} a-b & ad-ac \\ bc-bd & b-a \end{bmatrix}$. If $\mathbf{AB} - \mathbf{BA} = \mathbf{0}$ then $\mathbf{AB} = \mathbf{BA}$ which holds if either (i) $a = b = 0$, in which case $\mathbf{A}$ is the 2×2 unit matrix, or (ii) $a = b \neq 0$ and $c = d$, in which case both $\mathbf{A}$ and $\mathbf{B}$ are symmetric.

14. Under what conditions are the following matrix equalities true?

(*a*) $(\mathbf{X} + \mathbf{Y})^2 = \mathbf{X}^2 + 2\mathbf{XY} + \mathbf{Y}^2$.

(*b*) $(\mathbf{X} + \mathbf{Y})(\mathbf{X} - \mathbf{Y}) = \mathbf{X}^2 - \mathbf{Y}^2$.

Answer: After performing the operations it becomes clear that in both cases $\mathbf{XY} = \mathbf{YX}$ is needed.

15. Proove the following statements:

(*a*) Show that for any $m \times n$ matrix $\mathbf{A}$, both $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are symmetric. Give the dimensions of these matrices.

(*b*) Show that matrix $\mathbf{A}^T\mathbf{A}$ is positive semidefinite.

(*c*) Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be nonsingular matrices. Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

(*d*) Prove that $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$.

Answer: (*a*) Symmetry means: matrix is equal to its transpose, $\mathbf{X} = \mathbf{X}^T$. $\mathbf{A}^T\mathbf{A}$ is $n \times n$; $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A}$, thus it is symmetric. $\mathbf{AA}^T$ is $m \times m$; $(\mathbf{AA}^T)^T = (\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{AA}^T$, thus it is also symmetric., (*b*)Take any $\mathbf{x} \neq \mathbf{0}$. $\mathbf{x}^T(\mathbf{A}^T\mathbf{A})\mathbf{x} = (\mathbf{x}^T\mathbf{A}^T)(\mathbf{Ax}) = (\mathbf{Ax})^T(\mathbf{Ax}) = \|\mathbf{Ax}\|_2^2$. But $\|\mathbf{Ax}\|_2^2 \geq 0.$, (*c*) $(\mathbf{ABC})(\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{AB})(\mathbf{CC}^{-1})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$., (*d*) Using that $(\mathbf{XY})^T = \mathbf{Y}^T\mathbf{X}^T$, we can write $(\mathbf{ABC})^T = [(\mathbf{AB})\mathbf{C}]^T = \mathbf{C}^T(\mathbf{AB})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$.

16. How is the inverse of a matrix defined? Which matrices have an inverse? What are the main properties of the inverse?

Answer: Inverse is defined for square matrices. $\mathbf{A}^{-1}$ is defined to satisfy $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. If the columns of $\mathbf{A}$ are linearly independent then $\mathbf{A}^{-1}$ exists ($\mathbf{A}$ is nonsingular), otherwise it does not ($\mathbf{A}$ is singular). The inverse, if exists, is unique. The inverse of the inverse is the original matrix: $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

1.2 The linear programming problem

1. Convert the following linear programming constraints into equalities. Indicate the type of the associated logical variable. Try to combine constraints if possible.

$$2x_1 - 3x_2 + 4x_3 - 5x_4 \leq 6 \tag{1.1}$$
$$x_1 + x_2 - 3x_3 - x_4 \leq -6 \tag{1.2}$$
$$3x_1 + x_3 - x_4 \geq 2 \tag{1.3}$$
$$-2x_2 + 3x_4 \geq -1 \tag{1.4}$$
$$2x_3 + x_4 \leq 9 \tag{1.5}$$
$$2x_3 + x_4 \geq 4 \tag{1.6}$$
$$12 \geq 3x_1 - x_2 + x_3 + 2x_4 \geq 5 \tag{1.7}$$
$$x_1 + x_2 + x_3 + x_4 \bowtie 8 \tag{1.8}$$
$$x_1 + x_2 - x_3 - x_4 = 0 \tag{1.9}$$

Symbol $\bowtie$ indicates "nonbinding" (NB) constraints.

Answer:

Adding different logical variables to the constraints and noticing that constraints (1.5) and (1.6) can be combined into $4 \leq 2x_3 + x_4 \leq 9$, we obtain the following set of equality constraints:

$$\begin{aligned}
y_1 + 2x_1 - 3x_2 + 4x_3 - 5x_4 &= 6, & y_1 &\geq 0, & \text{type}(y_1) &= 2\\
y_2 + x_1 + x_2 - 3x_3 - x_4 &= -6, & y_2 &\geq 0, & \text{type}(y_2) &= 2\\
y_3 - 3x_1 - x_3 + x_4 &= -2, & y_3 &\geq 0, & \text{type}(y_3) &= 2\\
y_4 + 2x_2 - 3x_4 &= 1, & y_4 &\geq 0, & \text{type}(y_4) &= 2\\
y_5 + 2x_3 + x_4 &= 9, & 0 \leq y_5 &\leq 5, & \text{type}(y_5) &= 1\\
y_6 + 3x_1 - x_2 + x_3 + 2x_4 &= 12, & 0 \leq y_6 &\leq 7, & \text{type}(y_6) &= 1\\
y_7 + x_1 + x_2 + x_3 + x_4 &= 8, & y_7 &\text{ free}, & \text{type}(y_7) &= 3\\
y_8 + x_1 + x_2 - x_3 - x_4 &= 0, & y_8 &= 0, & \text{type}(y_8) &= 0.
\end{aligned}$$

2. Convert the following set of LP constraints to computational form #1. Indicate the

type of newly introduced variables (if any). Try to combine constraints if possible.

$$x_1 + 2x_3 \le 1 - x_4 \quad (1.10)$$
$$2x_1 + x_2 - 3x_3 - x_4 \le -1 \quad (1.11)$$
$$3x_1 + x_3 - x_4 \ge 0 \quad (1.12)$$
$$x_1 + 2x_3 + x_4 \ge -1 \quad (1.13)$$
$$9 \ge 2x_1 - x_2 + x_3 - 2x_4 \ge -1 \quad (1.14)$$
$$x_1 + x_2 + x_3 + x_4 \bowtie 0 \quad (1.15)$$
$$x_1 + x_2 - x_3 = x_4 \quad (1.16)$$

Symbol $\bowtie$ indicates "nonbinding" (NB) constraints.

Answer:

First, variables are moved to the left hand side (where appropriate). Then, adding different logical variables to the constraints and noticing that constraints (1.10) and (1.13) can be combined into a single range constraint, we obtain the following set of equality constraints:

$$\begin{array}{rll}
y_1 + x_1 + 2x_3 + x_4 = 1, & 0 \le y_1 \le 2, & \text{type}(y_1) = 1 \\
y_2 + 2x_1 + x_2 - 3x_3 - x_4 = -1, & y_2 \ge 0, & \text{type}(y_2) = 2 \\
y_3 - 3x_1 - x_3 + x_4 = 0, & y_3 \ge 0, & \text{type}(y_3) = 2 \\
y_4 + 2x_1 - x_2 + x_3 - 2x_4 = 9, & 0 \le y_4 \le 10, & \text{type}(y_4) = 1 \\
y_5 + x_1 + x_2 + x_3 + x_4 = 0, & y_5 \text{ free}, & \text{type}(y_5) = 3 \\
y_6 + x_1 + x_2 - x_3 - x_4 = 0, & y_6 = 0, & \text{type}(y_6) = 0
\end{array}$$

3. The following LP problem has two general constraints and four variables:

$$\begin{array}{rcl}
2x_1 - 3x_2 + 4x_3 - 5x_4 & \le & 6 \\
3x_1 - 4x_2 + 2x_3 + 2x_4 & \ge & 5
\end{array}$$

$$-1 \le x_1 \le 0,\ x_2 \ge 0,\ x_3 \le -2,\ x_4 \text{ free}.$$

Convert the joint constraint into equalities. Reverse minus type variables, if any, shift all finite lower bounds to zero. Indicate the type of newly created variables.

Answer:

First, convert into equalities:

$$\begin{array}{rcl}
y_1 + 2x_1 - 3x_2 + 4x_3 - 5x_4 & = & 6, \quad \text{type}(y_1) = 2, \\
y_2 - 3x_1 + 4x_2 - 2x_3 - 2x_4 & = & -5, \quad \text{type}(y_2) = 2, \\
-1 \le x_1 \le 0,\ x_2 \ge 0,\ x_3 & \le & -2,\ x_4 \text{ free}.
\end{array}$$

Next, reverse x_3 by defining $\bar{x}_3 = -x_3$ for which $\bar{x}_3 \geq 2$. We get

$$\begin{array}{rcl} y_1 + 2x_1 - 3x_2 - 4\bar{x}_3 - 5x_4 & = & 6, \\ y_2 - 3x_1 + 4x_2 + 2\bar{x}_3 - 2x_4 & = & -5, \\ -1 \leq x_1 \leq 0,\ x_2 \geq 0,\ \bar{x}_3 & \geq & 2,\ x_4 \text{ free.} \end{array}$$

Now, there are two variables with finite lower bound different from zero: x_1 and $\bar{x}_3$. We can define two translations: $\bar{x}_1 = x_1 - (-1)$, i.e., $\bar{x}_1 = x_1 + 1$ and $\bar{\bar{x}}_3 = \bar{x}_3 - 2$. Substituting $x_1 = \bar{x}_1 - 1$ and $\bar{x}_3 = \bar{\bar{x}}_3 + 2$ into the equations:

$$\begin{array}{rcl} y_1 + 2(\bar{x}_1 - 1) - 3x_2 - 4(\bar{\bar{x}}_3 + 2) - 5x_4 & = & 6, \\ y_2 - 3(\bar{x}_1 - 1) + 4x_2 + 2(\bar{\bar{x}}_3 + 2) - 2x_4 & = & -5, \\ 0 \leq \bar{x}_1 \leq 1,\ x_2 \geq 0,\ \bar{\bar{x}}_3 & \geq & 0,\ x_4 \text{ free,} \end{array}$$

which finally results in

$$\begin{array}{rcl} y_1 + 2\bar{x}_1 - 3x_2 - 4\bar{\bar{x}}_3 - 5x_4 & = & 16, \\ y_2 - 3\bar{x}_1 + 4x_2 + 2\bar{\bar{x}}_3 - 2x_4 & = & -12, \end{array}$$

$$0 \leq \bar{x}_1 \leq 1,\ x_2 \geq 0,\ \bar{\bar{x}}_3 \geq 0,\ x_4 \text{ free; type}(\bar{x}_1) = 1, \text{type}(\bar{\bar{x}}_3) = 2.$$

4. Consider the following linear programming problem:

$$\begin{array}{rrcrcrcr} \min & -2x_1 & + & 4x_2 & - & 12x_3 & & \\ \text{s.t.:} & -2x_1 & + & 4x_2 & - & 2x_3 & = & 0 \\ 4 \geq & x_1 & - & 3x_2 & + & x_3 & \geq & 12 \\ & 3x_1 & - & 6x_2 & + & 2x_3 & \geq & 5 \\ & \multicolumn{7}{c}{x_1 \geq -2,\ x_2 \geq 0,\ \ -1 \leq x_3 \leq 1} \end{array}$$

Convert the joint constraint into equalities. Reverse minus type variables, if any, shift all finite lower bounds to zero. Indicate the type of newly created variables. Don't forget to convert the objective function too.

Answer:

First, convert the constraints into equalities:

$$\begin{array}{rcll} y_1 - 2x_1 + 4x_2 - 2x_3 & = & 0, & \text{type}(y_1) = 0, \\ y_2 + x_1 - 3x_2 + x_3 & = & 12, & \text{type}(y_2) = 1,\ 0 \leq y_2 \leq 8, \\ y_3 - 3x_1 + 6x_2 - 2x_3 & = & 5, & \text{type}(y_3) = 2, \\ \multicolumn{4}{c}{-2 \leq x_1,\ 0 \leq x_2,\ -1 \leq x_3 \leq 1.} \end{array}$$

Now define $\bar{x}_1 = x_1 + 2$ and $\bar{x}_3 = x_3 + 1$ to shift the finite lower bonds to zero. This leads to:

$$\begin{array}{rcll}
\min\ -2(\bar{x}_1 - 2) + 4x_2 - 12(\bar{x}_3 - 1) & & & \\
y_1 - 2(\bar{x}_1 - 2) + 4x_2 - 2(\bar{x}_3 - 1) & = & 0, & \text{type}(y_1) = 0, \\
y_2 + (\bar{x}_1 - 2) - 3x_2 + (\bar{x}_3 - 1) & = & 12, & \text{type}(y_2) = 1, \\
y_3 - 3(\bar{x}_1 - 2) + 6x_2 - 2(\bar{x}_3 - 1) & = & 5, & \text{type}(y_3) = 2, \\
\multicolumn{4}{c}{0 \le \bar{x}_1,\ 0 \le x_2,\ 0 \le \bar{x}_3 \le 2,} \\
\multicolumn{4}{c}{0 = y_1,\ 0 \le y_2 \le 8,\ 0 \le y_3.}
\end{array}$$

which finally results in

$$\begin{array}{rcll}
\min\ -2\bar{x}_1 + 4x_2 - 12\bar{x}_3 & + & 16 & \\
y_1 - 2\bar{x}_1 + 4x_2 - 2\bar{x}_3 & = & -6, & \text{type}(y_1) = 0, \\
y_2 + \bar{x}_1 - 3x_2 + \bar{x}_3 - 1) & = & 3, & \text{type}(y_2) = 1, \\
y_3 - 3\bar{x}_1 + 6x_2 - 2\bar{x}_3 & = & -8, & \text{type}(y_3) = 2, \\
\multicolumn{4}{c}{0 \le \bar{x}_1,\ 0 \le x_2,\ 0 \le \bar{x}_3 \le 2,} \\
\multicolumn{4}{c}{0 = y_1,\ 0 \le y_2 \le 8,\ 0 \le y_3.}
\end{array}$$

5. What is the approximate number of potentially different bases for an LP problem with 27 constraints ($m = 27$) and 81 variables ($n = 81$).

   Hint: use the Stirling formula

$$k! \approx \sqrt{2\pi k}\ \left(\frac{k}{e}\right)^k.$$

   Take $\pi = 3.14$ and $e = 2.71$ (calculator needed).

   Answer:

   With $m = 27$ and $n = 81$ the number of potentially different bases is

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} = \binom{81}{27} = \frac{81!}{27!\,54!} \tag{1.17}$$

   Some details of computing $81!$:

$$\begin{aligned}
81! &\approx \sqrt{(2)(3.14)(81)}\left(\frac{81}{2.71}\right)^{81} \approx (22.51)\left(30^{81}\right) = (22.41)\left(10^{81}\right)\left(3^{81}\right) \\
&\approx (22.41)\left(10^{81}\right)(4.43)\left(10^{38}\right) \\
&\approx 10^{121}.
\end{aligned}$$

   In a similar fashion, $27! \approx 1.3 \times 10^{28}$ and $54! \approx 3.3 \times 10^{71}$. Continuing (1.17):

$$\frac{81!}{27!\,54!} \approx \frac{10^{121}}{(1.3)\left(10^{28}\right)(3.3)\left(10^{71}\right)} \approx 2.33 \times 10^{21}.$$

6. A chemical plant can produce 5 different types of fertilizer, F-1, ..., F-5. The production requires labour, energy, and processing on machines. These resources are available in limited amounts. The company wants to determine what quantities to produce that maximize the monthly revenue, assuming that any amount can be sold. The following table describes the technological requirements of producing one unit (tonne) of each product, the corresponding revenue and the monthly availability of the resources.

	F-1	F-2	F-3	F-4	F-5	Limit
Revenue	5	6	7	5	6	
Machine hour	2	3	2	1	1	1050
Labour hour	2	1	3	1	3	1050
Energy	1	2	1	4	1	1080

For instance, to produce one tonne of F-3 1 unit of energy is needed.

Formulate the linear programming model of the problem.

Answer:

The **decision variables** are the unknown quantities of the products. They are denoted by $x_1, \dots, x_5$.

The revenue to be maximized is $5x_1 + 6x_2 + 7x_3 + 5x_4 + 6x_5 + 7x_6$

The resource constraints are:

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 + 1x_4 + 1x_5 &\le 1050 \\ 2x_1 + 1x_2 + 3x_3 + 1x_4 + 3x_5 &\le 1050 \\ 1x_1 + 2x_2 + 1x_3 + 4x_4 + 1x_5 &\le 1080 \end{aligned}$$

Since production must be nonnegative, we impose $x_j \ge 0, \ j = 1, \dots, 5$.

7. A cattle farmer wants to minimize feeding costs while making sure the animals get the necessary weekly quantities of the four main nutrients. They are available in three stocks according to the following table.

	St-1	St-2	St-3	Required
Unit cost	8	9	7	
Nutr-1	4	3	2	600
Nutr-2	1	3	3	550
Nutr-3	2	2	0	400
Nutr-4	4	5	7	800

For instance, one unit of St-2 contains 2 units of nutrient 3. Column “Required” contains the minimum weekly requirements. One additional constraint is that no more than 300 units of St-1 is available per week.

Formulate the linear programming model of the problem.

Answer:

The decision variables are the unknown quantities of the stocks. They are denoted by x_1, x_2, x_3.

The costs to be minimized is $z = 8x_1 + 9x_2 + 7x_3$.

Constraints of the required quantities of nutrients are:

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 &\geq 600 \\ 1x_1 + 3x_2 + 3x_3 &\geq 550 \\ 2x_1 + 2x_2 + 0x_3 &\geq 400 \\ 4x_1 + 5x_2 + 7x_3 &\geq 800 \end{aligned}$$

Since the use of stocks must be nonnegative, we impose $x_j \geq 0,\ j = 1, \ldots, 3$ and, additionally, $x_1 \leq 300$.

8. A cargo company is preparing a ship with three stowages: front deck, rear deck and main stowage. Each stowage has a weight limit and a space capacity with the following limits:

Storage	Weight (t)	Space (m^3)
Front deck	10	10000
Rear deck	6	4500
Main stowage	20	8000

The following four cargoes are waiting to be shipped:

Cargo	Available quantity (t)	Volume (m^3/t)	Profit (\$/t)
C1	12	480	190
C2	10	550	220
C3	20	390	170
C4	16	600	250

Any proportion of the cargoes can be accepted if they are delivered. Formulate an optimization problem to maximize the profit of the delivery.

Answer:

Let the **decision variables** describe the amounts of cargoes assigned to the stowages. Let x_{ij} denote the the amount of the i^{th} cargo assigned to the j^{th} stowage unit (front deck, rear deck, main stowage). This way the weight and the space constraints can be given as:

$$x_{11} + x_{21} + x_{31} + x_{41} \leq 10$$
$$x_{12} + x_{22} + x_{32} + x_{42} \leq 6$$
$$x_{13} + x_{23} + x_{33} + x_{43} \leq 20$$
$$480x_{11} + 550x_{21} + 390x_{31} + 600x_{41} \leq 10000$$
$$480x_{12} + 550x_{22} + 390x_{32} + 600x_{42} \leq 4500$$
$$480x_{13} + 550x_{23} + 390x_{33} + 600x_{43} \leq 8000$$

The available quantities can be handled this way too:

$$x_{11} + x_{12} + x_{13} \leq 12$$
$$x_{21} + x_{22} + x_{23} \leq 10$$
$$x_{31} + x_{32} + x_{33} \leq 20$$
$$x_{41} + x_{42} + x_{43} \leq 16$$

Since the shipped quantities must be nonnegative, we impose $x_{ij} \geq 0$; $i = 1, \ldots, 4$; $j = 1, \ldots, 3$. The objective function describing the the profit is: $\max 190(x_{11} + x_{12} + x_{13}) + 220(x_{21} + x_{22} + x_{23}) + 170(x_{31} + x_{32} + x_{33}) + 250(x_{41} + x_{42} + x_{43})$.

9. The HR staff of a hospital would like to calculate the minimal number of nurses required for appropriate operation. The nurses are scheduled weekly in three shifts (6:00-14:00, 14:00-22:00, 22:00-06:00). The hospital needs nurses all the time, the minimum number of nurses required in a working week is given for each shift in the following table:

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Morning	8	9	7	11	8	6	5
Afternoon	9	7	8	10	4	3	3
Night	4	3	3	4	3	2	2

The work schedule of a nurse must meet the following requirements:

- Each nurse is assigned in the same shift (morning, afternoon or night) during a working period.
- The working period of a nurse consists of five consecutive days during any seven day long period.

Formulate an optimization problem that helps the HR staff determine the minimal number of nurses.

Answer:

Let the **decision variables** describe the number of nurses starting their working period on a given day and shift. Let x_{ij} denote the the number of nurses starting on the i^{th} day of the week in the j^{th} shift. For example x_{42} is the number of nurses starting their working period on Thursday afternoon. This way the number of nurses working on a given day and shift can be calculated from the work requirements. The constraints for Monday and Tuesday can be formulated as follows (for the other days the method of the formulation is similar):

$$\begin{aligned}
x_{41} + x_{51} + x_{61} + x_{71} + x_{11} &\leq 8 \\
x_{42} + x_{52} + x_{62} + x_{72} + x_{12} &\leq 9 \\
x_{43} + x_{53} + x_{63} + x_{73} + x_{13} &\leq 4 \\
x_{51} + x_{61} + x_{71} + x_{11} + x_{21} &\leq 9 \\
x_{52} + x_{62} + x_{72} + x_{12} + x_{22} &\leq 7 \\
x_{53} + x_{63} + x_{73} + x_{13} + x_{23} &\leq 3
\end{aligned}$$

The objective function is simply the sum of the variables $\min \sum_{i=1}^{7} \sum_{j=1}^{3} x_{ij}$. Don't forget to formulate the trivial non-negativity constraints as individual lower bounds $x_{ij};\ i = 1, \ldots, 7;\ j = 1, \ldots, 3$.

10. An electric company runs two coal-fired power plants, a new and an old one. These plants use different technologies. So, burning a ton of coal costs \$20 in the new plant or \$15 in the old plant. They produce 6150 kW/h or 5500 kW/h of electricity, respectively, while burning a ton of coal. There are three coal mines in the area. The monthly available amounts, supplier prices (\$) and the transportation costs (\$) are given in the following table:

Mine	Available amount (t)	Price $\$/t$
Expensive	400	70
Faraway	600	55
Fair	300	60

Also the transportation costs for a ton of coal are different depending on the physical locations of the mines and the plants:

	New plant	Old plant
Expensive	8\$	15\$
Faraway	30\$	25\$
Fair	12\$	13\$

Formulate a linear programming problem to maximize the monthly profit of the plants, if the selling price of 1 kW/h power is \$0.1 and each plant has a capacity limit of 500 tons for a month, assuming there is no loss in the produced amounts.

Answer:

Let the **decision variables** describe the amounts of coal bought by the plants. Let x_{11} and x_{12} be the amount bought from the mine called Expensive by the plants New and Old respectively. Similarly, x_{21}, x_{22} denote the quantities bought from the Faraway mine and x_{31}, x_{32} denote the amounts bought from the Fair mine. The constraints describing the available quantities and the plant capacities can be formulated as:

$$
\begin{aligned}
x_{11} + x_{12} &\leq 400 \\
x_{21} + x_{22} &\leq 600 \\
x_{31} + x_{32} &\leq 300 \\
x_{11} + x_{21} + x_{31} &\leq 500 \\
x_{12} + x_{22} + x_{32} &\leq 500
\end{aligned}
$$

All the other data given in the problem have to be considered in the objective function, which is: $\max\ 0.1(6150(x_{11}+x_{21}+x_{31})+5500(x_{21}+x_{22}+x_{23}))-20(x_{11}+x_{21}+x_{31})-15(x_{11}+x_{21}+x_{31})-78x_{11}-85x_{12}-85x_{21}-80x_{22}-72x_{31}-73x_{32}$

Finally, the trivial non-negativity constraints for the variables: $x_{ij} \geq 0;\ i = 1,2,3;\ j = 1,2.$

11. A company is considering three new products to replace current ones that are being discontinued. Management wants to determine which mix of these new products should be produced while observing three factors: long-run profit, stability of the workforce, and the level of capital investment in the new equipment. The goals in quantitative terms are: profit should be at least €125M, current level of employment of 4000 workers should be maintained, and the capital investment should not exceed €55M. Since goals may not be achievable, management decides to include the following penalties for the deviations. Penalty of 5 units for each €1M for missing the profit level; 2 units per 100 employees for going over employment goal and 4 units for going under the same goal; 3 units per €1M for exceeding the capital investment goal.

    It is assumed that the contribution of each new product to profit, employment and capital investment level is proportional to the rate of production (linearity assumption). The contributions per unit rate are the following:

	Contribution				Penalty
	P1	P2	P3	Goal	per unit
Long-run profit	12	9	15	at least 125 (in millions)	5
Employment level	5	3	4	exactly 40 (in hundreds)	2(+), 4(–)
Capital investment	5	7	8	at most 55 (in millions)	3

Set up a *goal programming* model for the problem. Hint: watch for the nature of the different goals ('at least', 'exactly', 'at most').

Answer:

Let the production levels of P1, P2 and P3 be denoted by $x_1,\ x_2$ and x_3. The goals can be stated as

$$\begin{array}{rrcl}
\text{Profit goal:} & 12x_1 + \ 9x_2 + \ 15x_3 & \geq & 125 \\
\text{Employment goal:} & 5x_1 + \ 3x_2 + \ 4x_3 & = & 40 \\
\text{Investment goal:} & 5x_1 + \ 7x_2 + \ 8x_3 & \leq & 55
\end{array}$$

Note, there is no overshoot penalty for profit and no undershoot penalty for investment. We introduce the following variables. Undershoot for profit, employment and investment: s_P, s_E, s_I and overshoot for the same: t_P, t_E, t_I. The goals now can be expressed as constraint

$$\begin{array}{rrrrrrrrrcr}
12x_1 + & 9x_2 + & 15x_3 & +s_P & -t_P & & & & & = & 125 \\
5x_1 + & 3x_2 + & 4x_3 & & & +s_E & -t_E & & & = & 40 \\
5x_1 + & 7x_2 + & 8x_3 & & & & & +s_I & -t_I & = & 55
\end{array}$$

with all variables nonnegative. The objective function is:

$$\min\ z = 5s_P + 4s_E + 2t_E + 3t_I$$

This is an LP problem that can be solved by standard methods. Note, t_P and s_I are not involved in the objective function.

12. The Father & Son haulage company is planning an extension of its fleet. Three types of trucks are included in the plan with the following characteristics:

Type	Load capacity (tons)	Cost (€1000)
Light	5	18
Medium	10	34
Large	20	55

Market analysis shows it would be desirable to add 10 light, 12 medium and 8 large models. The total capacity expansion should be around 300 tons and the total cost is limited to €1,000,000.

Write a *goal programming* model for the above problem if

- the financial constraint cannot be exceeded,
- it is equally undesirable to underachieve the number of light and medium models and overachieve the number of large models,
- it is undesirable to overachieve or underachieve the 300 ton goal of capacity expansion, underachievement being twice as bad as overachievement,

Explain your work.

Answer:

First, the decision variables have to be defined. They are the # of new trucks of capacities 5, 10, and 20 tons. The variables are denoted by $x_1,\ x_2$ and x_3, respectively; x_j-s are integers.

First of all, the financial constraint is "hard", as such it cannot be exceeded. However underachievement is allowed without penalty. Therefore,

$$18x_1 + 34x_2 + 55x_3 + s_1 = 1000 \quad \text{(in thousands)},$$

where s_1 is the deviational variable for underachievement (which, by the way, is not penalized).

Deviational variables for the total capacity expansion are denoted by $s_2,\ t_2$. Thus

$$5x_1 + 10x_2 + 20x_3 + s_2 - t_2 = 300.$$

Since the desired quantities of the trucks may not be achievable, nonnegative deviational variables have to be introduced: $s_3,\ s_4,\ s_5$ and $t_3,\ t_4,\ t_5$ for under and overachievement, respectively. Now we can write

$$\begin{aligned} x_1 + s_3 - t_3 &= 10 \\ x_2 + s_4 - t_4 &= 12 \\ x_3 + s_5 - t_5 &= 8 \end{aligned}$$

To minimize the "undesirabilities", we set up an objective function that penalizes the unwanted deviations. p_2^+ and p_2^- denote the penalties of over- and underachieving the total capacity by one ton with $p_2^- = 2p_2^+$ (underachievement is twice as 'costly'). p denotes the penalty of underachieving the number of light and medium, and overachieving the number of large models. So, the objective function is

$$\min\ 2p_2^- s_2 + p_2^- t_2 + ps_3 + ps_4 + pt_5$$

with respect to the five equality constraints above and all structural (x) and deviational (s and t) variables are nonnegative, structural variables are integers. Note, s_1 is not included in the objective function.

Chapter 2

Graphical solution of linear programming problems

2.1 Exercises

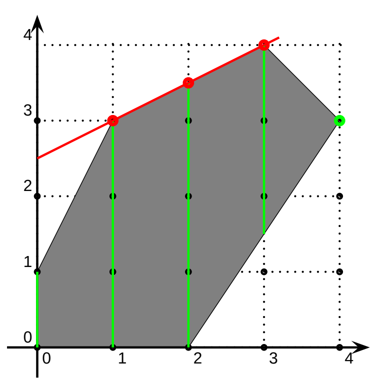
1. Solve the following integer programming problem graphically:

$$\max \; x_1 + x_2$$

subject to

$$\begin{array}{rcrcr} -10x_1 & + & 4x_2 & \leq & -3.0 \\ 2.5x_1 & + & x_2 & \leq & 6.75 \\ 5x_1 & - & 2x_2 & \leq & 7.5 \\ 2.5x_1 & + & x_2 & \geq & 3.75 \\ \multicolumn{5}{c}{0 \leq x_1, x_2 \leq 3 \text{ and integer.}} \end{array}$$

While the LP relaxation of the problem has an optimal solution, the feasible domain does not contain any point with all-integer coordinates, therefore, it is integer infeasible. The diagram below highlights the situation.



Chapter 3

The primal simplex algorithms: Phase I and Phase II

1. Determine the type of each variable in the following problem. Are the given solutions feasible? Do they satisfy the optimality conditions? Also, identify which solution is degenerate, if any, and say why.

   Problem: $\max\ \mathbf{c}^T\mathbf{x},\ \mathbf{Ax} = \mathbf{b}$,

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	$-\infty$	0	0	0
u_j	1	$+\infty$	$+\infty$	0	10	$+\infty$
type(x_j)						
In the solution:						
B/N	B	B	B	N	N	N
Value	1	1	-1	0	10	0
d_j	0	0	0	-10	10	0
Opt. cond. Y/N						

   Answer:

   The solution is feasible.

   The types of variables are $1, 2, 3, 0, 1, 2$. The optimality conditions are all satisfied (only nonbasic variables have to be considered) given this is a maximization problem.

   The solution is degenerate because basic variable $x_1 = 1$ which is its upper bound.

2. Determine the type of each variable in the following problem. Are the given solutions feasible? Do they satisfy the optimality conditions? Also, identify which solution is degenerate, if any, and say why.

Problem: $\min\ \mathbf{c}^T\mathbf{x},\ \mathbf{A}\mathbf{x} = \mathbf{b},$

	x_1	x_2	x_3	x_4	x_5	x_6
ℓ_j	0	0	0	$-\infty$	0	0
u_j	0	$+\infty$	$+\infty$	$+\infty$	10	$+\infty$
type(x_j)						
In the solution:						
B/N	B	B	B	N	N	N
Value	0	1	11	0	10	0
d_j	0	0	0	0	10	10
Opt. cond. Y/N						

Answer:

The solution is feasible but not optimal.

The types of variables are $0, 2, 2, 3, 1, 2$. Not all optimality conditions are satisfied: x_5 is of type-1, is nonbasic at upper bound, d_5 should be ≤ 0 given this is a minimization problem.

The solution is degenerate because basic variable $x_1 = 0$ which is its both lower and upper bound.

3. Show that the reduced cost of every basic variable is zero for any feasible basis $\mathbf{B}$ of a general linear programming problem.

   Answer:

   There are several ways to prove the statement. One of them is this. Let $\mathbf{a}_j$ the p-th basic variable. Then $c_{Bp} = c_j$ and $\mathbf{B}^{-1}\mathbf{a}_j = \mathbf{e}_p$ because $\mathbf{a}_j$ is basic in position p in the basis. By definition: $d_j = c_j - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{a}_j = c_j - \mathbf{c}_B^T\mathbf{e}_p = c_j - c_j = 0.$

4. Assume, we have an LP problem: $\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 11$ and a type-2 incoming variable, $x_q = 0$ with $d_q = -3$, are given. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. Is the BFS degenerate?

   Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_q^i	t_i
1	0	3	$+\infty$	1	
2	2	2	$+\infty$	-1	
3	3	1	4	1	
4	8	2	$+\infty$	2	
5	2	1	6	-2	
6	4	2	$+\infty$	1	

Answer:

The BSF is not degenerate despite of $x_{B1} = 0$. This is a type-3 variable, therefore, 0 is neither its lower nor its upper bound.

The ratios $\{-,\ -,\ 3,\ 4,\ 2,\ 4\}$ $\quad t_p = \min\{3, 4, 2, 4\} = 2$ with $p = 5$ (the row index that defines the minimum). $\theta = \min\{2,\ +\infty\} = 2$. Now, as $p = 5$ and $x_{B5} = 6$, x_{B5} leaves the basis at its upper bound of 6. x_q becomes the new x_{B5} with a value of $\hat{x}_q = x_q + \theta = 0 + 2 = 2$.

$\hat{\mathbf{x}}_B = \mathbf{x}_B(0) - \theta\boldsymbol{\alpha}_q$ which expands to

$$\hat{\mathbf{x}}_B = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 8 \\ 2 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \\ 4 \\ 6 \\ 2 \end{bmatrix} \quad \text{and } \mathbf{x}_{\hat{B}} = \begin{bmatrix} -2 \\ 4 \\ 1 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

The new BSF $\mathbf{x}_{\hat{B}}$ is not degenerate.

The new objective value $\hat{z} = z + \theta d_q = 11 + (2) \times (-3) = 5$.

5. Assume, we have an LP problem: $\max\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 10$ is given. A type-1 variable is coming in from its upper bound of 3. Its reduced cost is -2. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. Is the BFS degenerate?

   Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_q^i	t_i
1	0	3	$+\infty$	-1	
2	2	2	$+\infty$	1	
3	2	1	4	0	
4	8	2	$+\infty$	2	
5	0	0	0	0	

Answer:

This BSF is degenerate because $x_{B5} = 0$ which is its lower bound. **Note,** x_q is coming from its upper bound, its displacement is negative. No ratio is defined, x_q can be decreased by any value and the basic solution remains feasible.

However, type$(x_q) = 1$, therefore $-\theta = \min\{+\infty, 3\} = 3 \Rightarrow \ \theta = -3$. There is no basis change but bound swap: $\hat{x}_q = x_q + \theta = 3 + (-3) = 0$.

The new BFS is

$$\begin{bmatrix} 0 \\ 2 \\ 2 \\ 8 \\ 0 \end{bmatrix} - (-3) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 14 \\ 0 \end{bmatrix}$$

The solution remains degenerate. The new objective value $\hat{z} = z + \theta d_q = 10 + (-3) \times (-2) = 16$ (maximization problem!).

6. Assume, we have an LP problem: $\min\{\mathbf{c}^T\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$ and variables are subject to type specifications. A basic feasible solution (BFS) with $z = 3$ and a type-2 incoming variable, $x_q = 0$ with $d_q = -4$, are given. Also, $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ is available. Is the BFS degenerate?

   Determine the ratios, the value of the incoming variable, the variable leaving the basis (if any), the new BFS and the new value of the objective function. Is the new BFS degenerate?

i	x_{Bi}	type(x_{Bi})	u_{Bi}	α_q^i	t_i
1	2	2	$+\infty$	2	
2	2	2	$+\infty$	-1	
3	3	1	4	-1	
4	8	2	$+\infty$	2	
5	2	1	6	-2	

Answer:

The BFS is not degenerate. The ratios $\{1,\ -,\ 1,\ 4,\ 2\}$, $t_p = \min\{1, 1, 4, 2\} = 1$. Now p is either 1 or 3. Any of them can be chosen. In both cases, we have $\theta = \min\{1, +\infty\} = 1$.

The BFS after transformation is

$$\hat{\mathbf{x}}_B = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 8 \\ 2 \end{bmatrix} - 1 \times \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \\ 4 \end{bmatrix}$$

If $p = 1$ is chosen then x_{B1} leaves the basis at lower bound and x_q becomes the new x_{B1}. In case of $p = 3$, x_{B3} leaves the basis at upper bound and x_q will be the new x_{B3}. In both cases the value of x_q will be 1.

$$\text{If } p = 1 \quad \mathbf{x}_{\hat{B}} = \begin{bmatrix} 1 \\ 3 \\ \boxed{4} \\ 6 \\ 4 \end{bmatrix}; \quad \text{if } p = 3 \quad \mathbf{x}_{\hat{B}} = \begin{bmatrix} \boxed{0} \\ 3 \\ 1 \\ 6 \\ 4 \end{bmatrix}.$$

In either case the solution becomes degenerate. Degenerate positions are boxed. This example demonstrates how degeneracy is created.

The new objective value $\hat{z} = z + \theta d_q = 3 + (1) \times (-4) = -1$.

7. Solve the following linear programming problem using the simplex method with all types of variables

$$\begin{array}{rrcrcrcr}
\min\ z = & -4x_1 & - & 2x_2 & - & 12x_3 & & \\
\text{s.t.} & x_1 & + & 3x_2 & - & 2x_3 & \leq & 12 \\
& -3x_1 & + & x_2 & + & 2x_3 & = & 0 \\
-2 \leq & 4x_1 & - & x_2 & + & x_3 & \leq & 6 \\
\end{array}$$
$$x_1, x_2 \geq 0,\ \ 0 \leq x_3 \leq 1$$

Answer:

First, convert every constraint into equality by adding appropriate logical variables to them.

$$\begin{array}{rrcrcrcccl}
\min\ z = & -4x_1 & - & 2x_2 & - & 12x_3 & & & & \\
\text{s.t.} & x_1 & + & 3x_2 & - & 2x_3 & +y_1 & & & = \ 12 \\
& -3x_1 & + & x_2 & + & 2x_3 & & +y_2 & & = \ 0 \\
& 4x_1 & - & x_2 & + & x_3 & & & +y_3 & = \ 6 \\
\end{array}$$
$$x_1, x_2 \geq 0,\ \ 0 \leq x_3 \leq 1, \quad y_1 \geq 0, \quad y_2 = 0, \quad 0 \leq y_3 \leq 8$$

Starting basic feasible solution: $y_1 = 12$, $y_2 = 0$ and $y_3 = 6$. The problem in tabular form (UB is the column of upper bounds of basic variables):

	x_1	x_2	x_3	y_1	y_2	y_3	$\mathbf{x}_B$	UB
$\mathbf{u}$	∞	∞	1	∞	0	8		
y_1	1	3	-2	1			12	∞
y_2	-3	1	$\boxed{2}$		1		0	0
y_3	4	-1	1			1	6	8
$\mathbf{d}^T$	-4	-2	-12	0	0	0	0	
incoming: $x_3,\ \min\{\frac{0}{2}, \frac{6}{1}\} = 0,\ p = 2$								
y_1	-2	4	0	1	1	0	12	∞
x_3	$\boxed{-\frac{3}{2}}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	1
y_3	$\frac{11}{2}$	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	1	6	8
$\mathbf{d}^T$	-22	4	0	0	6	0	0	
incoming: $x_1,\ \min\{\frac{0-1}{-3/2}, \frac{6}{11/2}\} = \frac{2}{3},\ p = 2,\ x_3$ leaves at UB of 1								
	x_1	x_2	$\boxed{\boxed{x_3}}$	y_1	y_2	y_3	$\mathbf{x}_B$	UB
$\mathbf{u}$	∞	∞	1	∞	0	8		
y_1	0	$\boxed{\frac{10}{3}}$	$-\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{40}{3}$	∞
x_1	1	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0	$\frac{2}{3}$	∞
y_3	0	$\frac{1}{3}$	$\frac{11}{3}$	0	$\frac{4}{3}$	1	$\frac{7}{3}$	8
$\mathbf{d}^T$	0	$-\frac{10}{3}$	$-\frac{44}{3}$	0	$-\frac{4}{3}$	0	$\frac{44}{3}$	
incoming: $x_2,\ \min\{\frac{40/3}{10/3}, \frac{7/3}{1/3}\} = 4,\ p = 1$								
x_2	0	1	$-\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{10}$	0	4	∞
x_1	1	0	$-\frac{4}{5}$	$\frac{1}{10}$	$-\frac{3}{10}$	0	2	∞
y_3	0	0	$\frac{19}{5}$	$-\frac{1}{10}$	$\frac{13}{10}$	1	1	8
$\mathbf{d}^T$	0	0	-16	1	-1	0	28	

Optimal solution: $z = -28,\ x_1 = 2,\ x_2 = 4,\ x_3 = 1, y_3 = 1,\ y_1 = y_2 = 0.$

8. Let $\boldsymbol{\beta} = \mathbf{x}_B$ be a given infeasible basic solution and and type-2 incoming variable with the corresponding $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ column. Compute the phase-one reduced cost of the incoming variable, the value of the phase-one objective function, determine

the outgoing variable, the steplength, the new objective value and the updated x_B values

(a) using the traditional method,

(b) using the advanced method.

i	type(x_{Bi})	x_{Bi}	u_{Bi}	α_q^i
1	0	2	0	-1
2	1	5	1	1
3	2	-14	∞	-7
4	0	3	0	1
5	2	-4	∞	-2
6	3	-2	∞	-2

Answer:

As the incoming variable is a type-2 variable we are in the $t \leq 0$ case. First we have to evaluate the phase I objective function, which is $w = (-1-4)-(2+(5-1)+(3-0)) = -27$, and the phase I reduced cost, which is $d = (-7-4)-(-1+1+1) = -12$. After that we must determine the τ_{ℓ_i} and the τ_{u_i} breakpoints. Using the formulas shown in the problems part we get:

i	type(x_{Bi})	x_{Bi}	u_{Bi}	α_q^i	τ_{ℓ_i}	τ_{u_i}
1	0	2	0	-1	-	-
2	1	5	1	1	5	4
3	2	-14	∞	-7	2	-
4	0	3	0	1	3	3
5	2	-4	∞	-2	1	-
6	3	-2	∞	-2	-	-

(a) If we apply the traditional method, the smallest ratio determines the outgoing variable. This way we get that the displacement is $t = 1$. Applying $\hat{\mathbf{x}}_B = \mathbf{x}_B - t\boldsymbol{\alpha}_q$ and $\hat{w} = w - td$ we get the new values of the basic variables and the new objective value:

$$\hat{\mathbf{x}}_B = \begin{bmatrix} 2 \\ 5 \\ -14 \\ 3 \\ -4 \\ -2 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ -7 \\ 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -7 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{and } \mathbf{x}_{\hat{B}} = \begin{bmatrix} 3 \\ 4 \\ -7 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{w} = -27 - (1 \times -12) = -15$$

(b) To determine the outgoing variable and the steplength using the advanced method a table must be set up in which we are looking for the best objective value. The table is created by ordering the breakpoints:

k	r_k	$\alpha_q^{j_k}$	t_k	$w(t_k)$	j_k
0	-	-	0	-27	0
1	12	-4	1	-15	5
2	8	-7	2	-7	3
3	1	1	3	-6	4
4	0	1	3	-6	-4
5	-1	1	4	-7	-2
6	-2	1	5	-9	2

The table shows that the biggest improvement in the objective function can be achieved if the steplength is $t = 3$ ad the outgoing variable is x_{B_4}. It can go out from the basis on its lower bound or upper bound too, since it is a type-0 variable. Using this displacement the new objective value is $\hat{w} = -6$ and $x_{\hat{B}}$:

$$\hat{\mathbf{x}}_B = \begin{bmatrix} 2 \\ 5 \\ -14 \\ 3 \\ -4 \\ -2 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \\ -7 \\ 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 7 \\ 0 \\ 2 \\ 4 \end{bmatrix} \quad \text{and } \mathbf{x}_{\hat{B}} = \begin{bmatrix} 5 \\ 2 \\ 7 \\ 3 \\ 2 \\ 4 \end{bmatrix}$$

9. Let $\boldsymbol{\beta} = \mathbf{x}_B$ be a given infeasible basic solution and and type-1 incoming variable (its upper bound value is 6) with the corresponding $\boldsymbol{\alpha}_q = \mathbf{B}^{-1}\mathbf{a}_q$ column. Compute the phase-one reduced cost of the incoming variable, the value of the phase-one objective function, determine the outgoing variable, the steplength, the new objective value and the updated x_B values

   (a) using the traditional method,

   (b) using the advanced method.

i	type(x_{Bi})	x_{Bi}	u_{Bi}	α_q^i
1	1	-2	5	0
2	0	-2	0	-1
3	1	5	1	-1
4	1	3	1	1
5	1	-40	2	4
6	3	-4	∞	-2
7	2	-3	∞	3

Answer:

As the incoming variable is bounded and coming from its upper bound we are in the $t \leq 0$ case. First we have to evaluate the phase I objective function, which is $w = (-2-2-40-3) - ((5-1)+(3-1)) = -53$, and the phase I reduced cost, which is $d = (0-1+4+3) - (-1+1) = 6$. After that we must determine the $-\tau_{\ell_i}$ and the $-\tau_{u_i}$ breakpoints. Using the formulas shown in the problems part we get:

i	type(x_{Bi})	x_{Bi}	u_{Bi}	α_q^i	$-\tau_{\ell_i}$	$-\tau_{u_i}$
1	1	-2	5	0	-	-
2	0	-2	0	-1	-	-
3	1	5	1	-1	5	4
4	1	3	1	1	-	-
5	1	-40	2	4	10	-
6	3	-4	∞	-2	-	-
7	2	-3	∞	3	1	-

(a) If we apply the traditional method, the smallest ratio determines the outgoing variable. This way we get that the displacement is $t = 1$. Applying $\hat{\mathbf{x}}_B = \mathbf{x}_B - t\boldsymbol{\alpha}_q$ and $\hat{w} = w - td$ we get the new values of the basic variables and the new objective value:

$$\hat{\mathbf{x}}_B = \begin{bmatrix} -2 \\ -2 \\ 5 \\ 3 \\ -40 \\ -4 \\ -3 \end{bmatrix} - (-1) \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 4 \\ 4 \\ -36 \\ -6 \\ 0 \end{bmatrix} \quad \text{and } \mathbf{x}_{\hat{B}} = \begin{bmatrix} -2 \\ -2 \\ 4 \\ 4 \\ -36 \\ -6 \\ 1 \end{bmatrix}$$

$$\hat{w} = -53 - (-1 \times 6) = -47$$

(b) To determine the outgoing variable and the steplength using the advanced method a table must be set up in which we are looking for the best objective value. The table is created by ordering the breakpoints:

k	r_k	$\alpha_q^{j_k}$	t_k	$w(t_k)$	j_k
0	-	-	0	-53	0
1	6	3	1	-47	7
2	3	-1	4	-38	-3
3	2	-1	5	-36	3
4	1	4	10	-31	5

The table shows that the biggest improvement in the objective function can be achieved if the steplength is $t = 10$ ad the outgoing variable is x_{B_5}. There is an interesting situation here, because the computed steplength is bigger then the upper bound of the incoming variable. We have two options here. We can make a bound flip or further investigate the objective function. If the incoming variable enters the basis at an infeasible level, we have to consider it in the sum of infeasibilities. This means that if $t < 6$ it contributes in w with $-t - 6$, which means that the objective remains the same for $t = -6$ and $t = -10$, both gives $\hat{w} = -35$. This means that we can choose any t between -6 and -10. Since $t = -6$ can be achieved with a boundflip iteration, which is simpler than the basis change we choose that. Using this displacement the new objective value is $\hat{w} = -35$, the basis remains the same and $x_{\hat{B}}$:

$$\hat{\mathbf{x}}_B = \begin{bmatrix} -2 \\ -2 \\ 5 \\ 3 \\ -40 \\ -4 \\ -3 \end{bmatrix} - (-6) \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 4 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ -1 \\ 9 \\ -16 \\ -16 \\ 15 \end{bmatrix}$$

10. Find a feasible solution (if any) for the following linear programming problem.

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 2x_3 & \leq & -2 \\ 3x_1 & + & 2x_2 & - & 2x_3 & \geq & 6 \\ x_1 & & & - & x_3 & = & 3 \\ & & 2x_2 & - & x_3 & \leq & -1 \end{array}$$

$$x_1 \geq 0,\ x_2 \leq 0,\ 0 \leq x_3 \leq 1.$$

Answer:

First, convert every constraint into equality by adding appropriate logical variables to them.

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & + & y_1 & = & -2 \\ -3x_1 & + & 2x_2 & + & 2x_3 & + & y_2 & = & 6 \\ x_1 & & & - & x_3 & + & y_3 & = & 3 \\ & & -2x_2 & - & x_3 & + & y_4 & = & -1 \end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ 0 \leq x_3 \leq 1, y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.$$

Set up the first tableau and determine the phase I reduced costs:

	x_1	x_2	x_3	y_1	y_2	y_3	y_4	$\mathbf{x}_B$	UB
$\mathbf{u}$	∞	∞	1	∞	∞	0	∞		
y_1	1	3	2	1	0	0	0	-2	∞
y_2	-3	2	2	0	1	0	0	-6	∞
y_3	1	0	-1	0	0	1	0	3	0
y_4	0	-2	-1	0	0	0	1	1	∞
$\mathbf{d}^T$	-3	5	5	–	–	–	–	-11	

Since all the x_i variables are at their lower bounds, an appropriate displacement should be negative ($t < 0$). Such a displacement is defined by x_1, it will be the candidate to enter the basis. The only ratio definined by the phase I ratio test defines the outgoing variable y_2. After the transformation we get:

	x_1	x_2	x_3	y_1	y_2	y_3	y_4	$\mathbf{x}_B$	UB
$\mathbf{u}$	∞	∞	1	∞	∞	0	∞		
y_1	0	$-\frac{11}{3}$	$\frac{8}{3}$	1	$\frac{1}{3}$	0	0	-4	∞
x_1	1	$-\frac{2}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0	0	2	∞
y_3	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	0	1	0
y_4	0	-2	-1	0	0	0	1	1	∞
$\mathbf{d}^T$	–	3	3	–	0	–	–	-5	

Here we can see that there is no improving candidate but the current solution is infeasible, thus there is no feasible solution for the problem.

11. Solve the following linear programming problems using the two phase primal simplex method.

$$
\begin{array}{llrcrcrcr}
\max & z = & -x_1 & + & 2x_2 & - & x_3 & & \\
\text{s.t.} & & 2x_1 & - & 2x_2 & + & x_3 & = & 6 \\
 & & 3x_1 & - & 5x_2 & + & 2x_3 & \leq & 15 \\
 & & x_1 & + & x_2 & - & x_3 & \geq & -3 \\
 & & -x_1 & + & 3x_2 & - & x_3 & \leq & -1
\end{array}
$$

$$x_1 \geq 0,\ x_2 \text{ free},\ 0 \leq x_3 \leq 2.$$

Answer:

First, convert every constraint into equality by adding appropriate logical variables to them.

$$
\begin{array}{llrcrcrcrcr}
\min & -z = & x_1 & - & 2x_2 & + & x_3 & & & & \\
\text{s.t.} & & 2x_1 & - & 2x_2 & + & x_3 & + & y_1 & = & 6 \\
 & & 3x_1 & - & 5x_2 & + & 2x_3 & + & y_2 & = & 15 \\
 & & -x_1 & - & x_2 & + & x_3 & + & y_3 & = & 3 \\
 & & -x_1 & + & 3x_2 & - & x_3 & + & y_4 & = & -1
\end{array}
$$

$$x_1 \geq 0,\ x_2 \text{ free},\ 0 \leq x_3 \leq 2, y_1 \geq 1, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0.$$

First, we have to set up the tableau. The initial solution is infeasible, so we have to compute the phase I reduced costs and perform an iteration based on the phase I ratio test. If we apply the advanced method during the ratio test we immediatly get a basic feasile solution. After that we have to be aware that x_2 is a free variable, so it will be a candidate to enter the basis. After this transtormation we get the optimal solution. The tableaux are the following:

	x_1	x_2	x_3	y_1	y_2	y_3	y_4	$\mathbf{x}_B$	UB
$\mathbf{u}$	∞	∞	2	0	∞	∞	∞		
y_1	$\boxed{2}$	-2	1	1	0	0	0	6	0
y_2	3	-5	2	0	1	0	0	15	∞
y_3	-1	-1	1	0	0	1	0	3	∞
y_4	-1	3	-1	0	0	0	1	-1	∞
$\mathbf{d}^T$	1	-2	-1	0	0	0	0	0	
$\mathbf{d}_I^T$	-3	5	-2	$-$	$-$	$-$	$-$	-7	
x_1	1	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	3	∞
y_2	0	-2	$\frac{1}{2}$	$-\frac{3}{2}$	1	0	0	6	∞
y_3	0	-2	$\frac{3}{2}$	$\frac{1}{2}$	0	1	0	6	∞
y_4	0	$\boxed{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1	2	∞
$\mathbf{d}^T$	0	-1	$\frac{1}{2}$	$\frac{3}{2}$	0	0	0	-3	
x_1	1	0	$\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{1}{2}$	4	∞
y_2	0	0	$-\frac{3}{2}$	-2	1	0	1	8	∞
y_3	0	0	$-\frac{1}{2}$	1	0	1	1	8	∞
x_2	0	1	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$	1	∞
$\mathbf{d}^T$	0	0	$\frac{1}{4}$	$\frac{7}{4}$	0	0	$\frac{1}{2}$	-2	

Optimal solution: $z = -2,\ x_1 = 4,\ x_2 = 1,\ y_2 = 8,\ y_3 = 8$, and $x_3 = y_1 = y_2 = 0$.

12. Solve the following linear programming problems using the two phase primal simplex method.

$$
\begin{array}{llrcrcr}
\min & z = & -x_1 & + & x_2 & & \\
\text{s.t.} & & 2x_1 & + & x_2 & \geq & 1 \\
& & x_1 & + & x_2 & \leq & 3 \\
& & x_1 & - & x_2 & \geq & -1 \\
& & \multicolumn{5}{l}{x_1, x_2 \geq 0}
\end{array}
$$

Answer:

First, convert every constraint into equality by adding appropriate logical variables to them.

$$\begin{array}{llrcrcrcr}
\min & -z = & -x_1 & + & x_2 & & & & \\
\text{s.t.} & & -2x_1 & - & x_2 & + & y_1 & = & -1 \\
 & & x_1 & + & x_2 & + & y_2 & = & 3 \\
 & & -x_1 & + & x_2 & + & y_3 & = & 1 \\
 & & \multicolumn{7}{l}{x_1, x_2, y_1, y_2, y_3 \geq 0.}
\end{array}$$

First, we have to set up the tableau. The initial solution is infeasible, so we have to compute the phase I reduced costs and perform an iteration based on the phase I ratio test. The traditional and the advanced method also reach a feasible solution in one iteration, we are using the smallest ration in the computation now. After that only one phase II itaeration is necessary to reach optimality. The tableaux are the following:

	x_1	x_2	y_1	y_2	y_3	$\mathbf{x}_B$
y_1	$\boxed{-2}$	-1	1	0	0	-1
y_2	1	1	0	1	0	3
y_3	-1	1	0	0	1	1
$\mathbf{d}^T$	1	1	0	0	0	0
$\mathbf{d}_I^T$	-2	-1	$-$	$-$	$-$	-1
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
y_2	0	$-\frac{1}{2}$	$\boxed{\frac{1}{2}}$	1	0	$\frac{5}{2}$
y_3	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	1	$\frac{3}{2}$
$\mathbf{d}^T$	0	$\frac{3}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
x_1	1	0	0	1	0	3
y_1	0	1	1	2	0	5
y_3	0	1	0	1	1	4
$\mathbf{d}^T$	0	1	0	1	0	3

Optimal solution: $z = -2,\ x_1 = 2,\ x_2 = 0,\ y_1 = 5,\ y_2 = 0, y_3 = 4$

Chapter 4

The dual simplex algorithms: Phase I and Phase II

1. Formulate the dual of the following problem.

$$
\begin{array}{rrrrrcr}
\max z = & x_1 - & 2x_2 + & 3x_3 - & 4x_4 & & \\
\text{s.t.} & 2x_1 + & 5x_2 - & 4x_3 + & 9x_4 & \leq & 15 \\
 & x_1 + & 4x_2 + & 2x_3 - & 6x_4 & \geq & -7 \\
 & 4x_1 - & 3x_2 - & 6x_3 + & 4x_4 & = & 1 \\
 & & x_1, \ldots, x_4 \geq 0. & & & &
\end{array}
$$

Answer:

$$
\begin{array}{rrrrcr}
\min & 15y_1 - & 7y_2 + & y_3 & & \\
\text{s.t.} & 2y_1 + & y_2 + & 4y_3 & \geq & 1 \\
 & 5y_1 + & 4y_2 - & 3y_3 & \geq & -2 \\
 & -4y_1 + & 2y_2 - & 6y_3 & \geq & 3 \\
 & 9y_1 - & 6y_2 + & 4y_3 & \geq & -4 \\
 & & y_1 \geq 0, \ y_2 \leq 0, \ y_3 \text{ free.} & & &
\end{array}
$$

2. Write the dual of the problem given by

$$
\begin{array}{rrrrcr}
\min z = & -x_1 + & 2x_2 + & 6x_3 & & \\
\text{s.t.} & x_1 + & 3x_2 - & 2x_3 & \geq & 0 \\
 & -x_1 - & 2x_2 + & 5x_3 & = & 0 \\
 & 2x_1 + & 3x_2 + & 4x_3 & \leq & 0 \\
 & & x_1 \leq 0, \ x_2, x_3 \geq 0. & & &
\end{array}
$$

Answer:

There is no objective function here because all coefficients are zero. So, the dual reduces to a feasibility problem.

Find a feasible solution to

$$\begin{array}{rrrcr} y_1 - & y_2 + & 2y_3 & \geq & -1 \\ 3y_1 - & 2y_2 + & 3y_3 & \leq & 2 \\ -2y_1 + & 5y_2 + & 4y_3 & \leq & 6 \\ \multicolumn{5}{l}{y_1 \geq 0,\ y_2 \text{ free}, y_3 \leq 0.} \end{array}$$

3. Formulate the dual of the following linear programming problem

$$\begin{array}{lll} \text{(P)} & \min & \mathbf{c}^T\mathbf{x} \\ & \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \\ & & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Show that the dual of the dual is the original primal problem (P).

Answer:

The dual of (P) is

$$\begin{array}{llclr} (D) & \max\ \mathbf{b}^T\mathbf{y} & & & -\min\,(-\mathbf{b}^T\mathbf{y}) \\ s.t. & \mathbf{A}^T\mathbf{y} \leq \mathbf{c} & \text{which can be written as} & & -\mathbf{A}^T\mathbf{y} \geq -\mathbf{c} \\ & \mathbf{y} \geq \mathbf{0} & & & \mathbf{y} \geq \mathbf{0} \end{array}$$

This is the form of (P). Therefore, we can apply the method of writing its dual and obtain

$$\begin{array}{rcr} -\max\,(-\mathbf{c}^T\mathbf{x}) & & \min\ \mathbf{c}^T\mathbf{x} \\ -\mathbf{A}\mathbf{x} \geq -\mathbf{b} & \text{that is equivalent to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} & & \mathbf{x} \geq \mathbf{0} \end{array}$$

which is the primal.

4. The weak duality theorem says that if an arbitrary primal feasible solution $\mathbf{x}$ is given than for any dual feasible solution $\mathbf{y}$ the relation $\mathbf{b}^T\mathbf{y} \leq \mathbf{c}^T\mathbf{x}$ holds. The strong duality theorem says that if a problem has a feasible finite solution then its dual pair has a feasible finite optimum too, and the objective values are the same. Prove these theorems.

Answer: First we start with the weak duality theorem:

Let's assume that the primal problem is in standard from, which implies that $\mathbf{x} \geq 0$ and the $\mathbf{y}$ variables are free. From the primal problem we know that $\mathbf{b} = \mathbf{A}\mathbf{x}$ because $\mathbf{x}$ is a primal solution. We can multiply this equation from the left by $\mathbf{y}^T$ because it is a vector of free variables, which leads to $\mathbf{y}^T\mathbf{b} = \mathbf{y}^T\mathbf{A}\mathbf{x}$. For a dual feasible solution the $\mathbf{y}^T\mathbf{A} \leq \mathbf{c}$ inequality must hold. If we apply this inequality to the previous equation (remember, $\mathbf{x} \geq 0$ we will have $\mathbf{y}^T\mathbf{b} = \mathbf{y}^T\mathbf{A}\mathbf{x} \leq \mathbf{c}\mathbf{x}$, which is exactly the inequality we wold like to proof.

Proving strong duality:

Let's assume the standard form again. If the primal problem has a feasible finite solution then an optimal basis $\mathbf{B}$ can be found. The optimal solution is $\mathbf{x}^* = \mathbf{B}^{-1}\mathbf{b}$. If the solution is optimal, all the reduced costs satisfy the optimality conditions, so $\mathbf{c}^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A} \geq 0$. If we denote $\mathbf{c}_B^T\mathbf{B}^{-1}$ as $\mathbf{y}^T$ and transform the equation we get $\mathbf{y}^T\mathbf{A} = \mathbf{c}$,which means that $\mathbf{y}$ is a dual feasible solution. To prove the theorem we have to check the equality of the objective values too: $z = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} = \mathbf{y}^T\mathbf{b}$. This means that the two objective values are the same, strong duality holds.

5. Show that if the primal has an unbounded solution the dual problem has no feasible solution (the dual problem is infeasible). Use the following primal-dual pair:

$$\begin{array}{rl}
(P) & \min\ \mathbf{c}^T\mathbf{x} \\
s.t. & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & \mathbf{x} \geq \mathbf{0}
\end{array}
\qquad\qquad
\begin{array}{rl}
(D) & \max\ \mathbf{b}^T\mathbf{y} \\
s.t. & \mathbf{A}^T\mathbf{y} \leq \mathbf{c}
\end{array}$$

Answer:

Assume the primal has an unbounded solution and the dual still has a feasible solution, denoted by $\mathbf{y}$. This will lead to a contradiction. According to the *weak duality theorem*, for every primal feasible solution the $\mathbf{y}^T\mathbf{b} \leq \mathbf{c}^T\mathbf{x}$ relation holds. However, it is impossible as $\mathbf{c}^T\mathbf{x} \to -\infty$. Therefore, such a $\mathbf{y}$ cannot exists, the dual has no feasible solution.

6. Investigate whether the following linear programming problem can be solved with the dual simplex method. Explain your answer. If yes, convert the problem into a form needed by the algorithm and solve it. Discuss the solution steps in detail. Provide solutions for both the primal and dual.

$$\begin{array}{lrcrcrcrcr}
\text{max} & x_1 & - & 4x_2 & - & 2x_3 & - & 2x_4 & & \\
\text{s. t.} & -2x_1 & - & x_2 & - & x_3 & & & \leq & -1 \\
 & 2x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 3 \\
 & & & 4x_2 & - & x_3 & - & 2x_4 & \leq & -2 \\
 & \multicolumn{9}{c}{x_1 \leq 0,\ x_j \geq 0,\ j = 2, \ldots, 4}
\end{array}$$

Answer:

The problem is suitable for the dual simplex algorithm if some simple transformations are performed: (i) convert $x_1 \leq 0$ to $x_1 \geq 0$ by reversing the sign of the coefficients of x_1 in the objective function and in the constraints (and remember to undo it in the final solution), (ii) convert the problem to minimization by reversing the signs of the objective coefficients (and remember it when declaring the optimal solution). The resulting problem is

$$
\begin{array}{lrcrcrcrcr}
\min & x_1 & + & 4x_2 & + & 2x_3 & + & 2x_4 & & \\
\text{s. t.} & 2x_1 & - & x_2 & - & x_3 & & & \leq & -1 \\
 & -2x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 3 \\
 & & & 4x_2 & - & x_3 & - & 2x_4 & \leq & -2 \\
 & \multicolumn{9}{c}{x_j \geq 0, \ j = 1, \ldots, 4}
\end{array}
$$

After adding type-2 logical variables to each constraint ($x_5, \ x_6$ and x_7, respectively) they become equalities so that the dual simplex can commence.

The bottom row contains the dual logical variables. Pivot elements of the next iteration are boxed. The following steps are performed.

B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{x}_B$
x_5	2	-1	-1	0	1			-1
x_6	-2	1	1	1		1		3
x_7	0	4	-1	$\boxed{-2}$			1	-2
$\mathbf{w}^T$	1	4	2	2	0	0	0	0

B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{x}_B$
x_5	2	-1	$\boxed{-1}$	0	1			-1
x_6	-2	3	$1/2$	0	0	1	$1/2$	2
x_4	0	-2	$1/2$	1	0	0	$-1/2$	1
$\mathbf{w}^T$	1	8	1	0	0	0	1	-2

The next tableau

B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{x}_B$
x_3	-2	1	1	0	-1	0	0	1
x_6	-1	$5/2$	0	0	$1/2$	1	$1/2$	$3/2$
x_4	1	$-5/2$	0	1	$1/2$	0	$-1/2$	$1/2$
$\mathbf{w}^T$	3	7	0	0	1	0	1	-3

is primal (and dual) feasible, thus optimal. The optimal basis is $\mathcal{B} = \{3, 6, 4\}$ The optimal solution is $x_1 = x_2 = x_5 = x_7 = 0$, $x_3 = 1, \ x_4 = 0.5$ and $x_6 = 1.5$. The value of the objective function is -3.

The dual solution is obtained from its definition of $\mathbf{y}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$. The inverse of the basis $\mathbf{B}^{-1}$ can found in the final tableau just above the logical variables, while $\mathbf{c}_B = [-2, 0, -2]^T$. Therefore, the dual solution is $\mathbf{y}^T = [1, 0, 1]^T$ which gives a dual objective value of -3, the same as the primal ($\mathbf{c}_B^T \mathbf{x}_B = \mathbf{b}^T \mathbf{y}$).

7. Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{lrcrcrcrcr}
\min & x_1 & - & 2x_2 & + & 4x_3 & + & 4x_4 & & \\
\text{s. t.} & 2x_1 & + & 4x_2 & - & 4x_3 & & & \leq & -1 \\
 & x_1 & + & 4x_2 & & & + & 2x_4 & \geq & 2 \\
 & x_1 & - & x_2 & + & x_3 & + & x_4 & \leq & 3 \\
 & \multicolumn{9}{c}{x_j \geq 0,\ j = 1, 3, 4,\ x_2 \leq 0.}
\end{array}$$

Answer:

First, x_2 is replaced by its negative to make it a type-2 variable. This is achieved by reversing the sign of every coefficient multiplying it.

Next, the second constraint ($\geq$) is multiplied by -1 to make it $\leq$. The resulting problem becomes

$$\begin{array}{lrcrcrcrcr}
\min & x_1 & + & 2x_2 & + & 4x_3 & + & 4x_4 & & \\
\text{s. t.} & 2x_1 & - & 4x_2 & - & 4x_3 & & & \leq & -1 \\
 & -x_1 & + & 4x_2 & & & - & 2x_4 & \leq & -2 \\
 & x_1 & + & x_2 & + & x_3 & + & x_4 & \leq & 3 \\
 & \multicolumn{9}{c}{x_j \geq 0,\ j = 1, 2, 3, 4.}
\end{array}$$

After adding a type-2 logical variable s_i to each row, the following tableau can be set up:

B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	$\mathbf{x}_B$
s_1	2	-4	-4	0	1			-1
s_2	$\boxed{-1}$	4	0	-2		1		-2
s_3	1	1	1	1			1	3
d_j	1	2	4	4	0	0	0	0
s_1	0	4	-4	$\boxed{-4}$	1	2		-5
x_1	1	-4	0	2	0	-1		2
s_3	0	5	1	-1	0	1	1	1
d_j	0	6	4	2	0	1	0	-2
x_4	0	-1	1	1	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{5}{4}$
x_1	1	-2	$\boxed{-2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$
s_3	0	4	2	0	$-\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{9}{4}$
d_j	0	8	2	0	$\frac{1}{2}$	2	0	$-\frac{9}{2}$
x_4	$\frac{1}{2}$	-2	0	1	0	$-\frac{1}{2}$	0	1
x_3	$-\frac{1}{2}$	1	1	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$
s_3	1	2	0	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{7}{4}$
d_j	1	6	0	0	1	2	0	-5

Thus the solution is $x_3 = 1/4$, $x_4 = 1$, $s_3 = 7/4$ and $x_1 = x_2 = s_1 = s_2 = 0$. The value of the objective function is 5.

8. Solve the following linear programming problem using the dual simplex method.

$$
\begin{array}{llrcrcrcrcl}
\text{min} & z = & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & & \\
\text{s.t.} & & x_1 & & & & & + & x_4 & \geq & 4 \\
 & & x_1 & + & x_2 & & & & & \geq & 8 \\
 & & & & x_2 & + & x_3 & & & \geq & 8 \\
 & & & & & & x_3 & + & x_4 & \geq & 6 \\
\end{array}
$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Answer:

First, multiply all the constraints by -1 to make them <= constraints and add type-2 logical variables (y_i) to all of them. The resulting problem is in CF-1:

$$
\begin{array}{llrcrcrcrcrcl}
\text{min} & z = & x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & & & & \\
\text{s.t.} & & -x_1 & & & & & - & x_4 & + & y_1 & = & 4 \\
 & & -x_1 & - & x_2 & & & & & + & y_2 & = & 8 \\
 & & & - & x_2 & - & x_3 & & & + & y_3 & = & 8 \\
 & & & & & - & x_3 & - & x_4 & + & y_4 & = & 6 \\
\end{array}
$$
$$x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \geq 0$$

The following tableau can be set up:

B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	$\mathbf{x}_B$
s_1	-1	0	0	-1	1	0	0	0	-4
s_2	$\boxed{-1}$	-1	0	0	0	1	0	0	-8
s_3	0	-1	-1	0	0	0	1	0	-8
s_4	0	0	-1	-1	0	0	0	1	-6
d_j	1	2	3	4	0	0	0	0	0
s_1	0	1	0	-1	1	-1	0	0	4
s_2	1	1	0	0	0	-1	0	0	-8
s_3	0	$\boxed{-1}$	-1	0	0	0	1	0	-8
s_4	0	0	-1	-1	0	0	0	1	-6
d_j	0	1	3	4	0	1	0	0	-8
s_1	0	0	-1	-1	-1	0	1	0	-4
s_2	0	1	-1	0	-1	0	1	0	0
s_3	0	1	1	0	0	0	-1	0	8
s_4	0	0	$\boxed{-1}$	-1	0	0	0	1	-6
d_j	0	0	2	4	1	0	1	0	-16

So the final tableau is:

B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	s_4	$\mathbf{x}_B$
s_1	0	0	0	0	-1	0	1	-1	2
s_2	1	0	0	1	-1	0	1	-1	6
s_3	0	1	0	-1	0	0	-1	1	2
s_4	0	0	1	1	0	0	0	1	6
d_j	0	0	0	2	1	0	1	2	-28

Thus the solution is $x_1 = 6$, $x_2 = 2$, $x_3 = 6$, $s_1 = 2$ and $x_4 = s_2 = s_3 = s_4 = 0$. The value of the objective function is 28.

9. Solve the following primal-dual pair with using the primal and the dual simplex algorithms. Verify that the strong duality theorem holds for the solutions:

$$
\begin{array}{llrcrcrcr}
\text{max} & z = & & & -x_2 & - & 2x_3 & & \\
\text{s.t.} & & x_1 & + & x_2 & - & 2x_3 & \leq & -2 \\
 & & 2x_1 & - & x_2 & + & x_3 & \leq & -4 \\
\end{array}
$$
$$x_1, x_2, x_3 \geq 0.$$

$$
\begin{array}{llrcrcr}
\text{min} & w = & -2y_1 & - & 4y_2 & & \\
\text{s.t.} & & y_1 & + & 2y_2 & \geq & 0 \\
 & & y_1 & - & y_2 & \geq & -1 \\
 & & -2y_1 & + & y_2 & \geq & -2 \\
\end{array}
$$
$$y_1, y_2 \geq 0.$$

Answer:

First, investigate the structure of the primal dual pair. If we construct the CF-1 for both problems, we will get the following form:

$$
\begin{array}{llrcrcrcrcr}
\text{min} & -z = & & & x_2 & + & 2x_3 & & & & \\
\text{s.t.} & & x_1 & + & x_2 & - & 2x_3 & + & s_1 & = & -2 \\
 & & 2x_1 & - & x_2 & + & x_3 & + & s_2 & = & -4 \\
\end{array}
$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0.$$

$$
\begin{array}{llrcrcrcr}
\text{min} & w = & -2y_1 & - & 4y_2 & & & & \\
\text{s.t.} & & -y_1 & - & 2y_2 & + & d_1 & = & 0 \\
 & & -y_1 & + & y_2 & + & d_2 & = & 1 \\
 & & 2y_1 & - & y_2 & + & d_3 & = & 2 \\
\end{array}
$$
$$y_1, y_2, d_1, d_2, d_3 \geq 0.$$

The first problem is suitable to solve with the dual algorithm, since the right hand side vector contains negative elements, and the second one is suitable for the primal algorithm.

Let's start with the first one, which gives the following tableau:

B	x_1	x_2	x_3	s_1	s_2	$\mathbf{x}_B$
s_1	1	1	-2	1	0	-2
s_2	2	$\boxed{-1}$	1	0	1	-4
d_j	0	1	2	0	0	0
s_1	3	0	$\boxed{-1}$	1	1	-6
x_1	-2	1	-1	0	-1	4
d_j	2	0	3	0	1	-4

So the final tableau is:

B	x_1	x_2	x_3	s_1	s_2	$\mathbf{x}_B$
x_2	-3	0	1	-1	-1	6
x_1	1	1	0	1	0	10
d_j	11	0	0	3	4	-22

This gives a solution with $x_1 = 10$, $x_2 = 6$, $x_3 = 0$, $s_0 = 1$, $s_2 = 0$ and the objective value is $z = -22$ (note that we multiplied the objective function by -1 in the first step). We can also read the dual solution from the final tableau (from the reduced costs of the logical variables), we expect the dual variables to be $y_1 = 3$ and $y_2 = 4$ from the dual problem. Let's solve it with the primal simplex algorithm. The tableaux during computation are the following:

B	y_1	y_2	d_1	d_2	d_3	$\mathbf{x}_B$
d_1	-1	-2	1	0	0	0
d_2	-1	$\boxed{1}$	0	1	0	1
d_3	2	-1	0	0	1	2
w_j	-2	-4	0	0	0	0
d_1	-3	0	1	-1	0	0
y_2	-1	1	0	1	0	1
d_3	$\boxed{1}$	0	0	2	1	2
w_j	-6	0	0	-2	0	4

So the final tableau is:

B	x_1	x_2	x_3	s_1	s_2	$\mathbf{x}_B$
d_1	0	0	1	5	3	11
y_2	0	1	0	-1	1	4
y_1	1	0	0	2	1	3
w_j	0	0	0	10	6	22

Thus the solution is $y_1 = 3$, $y_2 = 4$, $d_1 = 11$, $d_2 = 0$, $d_3 = 0$, which is exactly the same as what we expected. Also the optimal solution verifies strong duality, both problems gave -22 as the objective value.

10. Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{lrrrrrrrrrr} \max & z = & -2x_1 & - & x_2 & - & 3x_3 & - & x_4 & & \\ \text{s.t.} & & -x_1 & + & 2x_2 & + & x_3 & - & x_4 & \leq & 3 \\ & & -x_1 & - & x_2 & + & x_3 & + & 2x_4 & \geq & 2 \\ & & x_1 & - & 2x_2 & - & 3x_3 & - & x_4 & \leq & -2 \end{array}$$

$$x_1, \ldots, x_4 \geq 0.$$

Answer:

First convert the problem to CF-1:

$$\begin{array}{lrrrrrrrrrrrr} \min & -z = & 2x_1 & + & x_2 & + & 3x_3 & + & x_4 & & & & \\ \text{s.t.} & & -x_1 & + & 2x_2 & + & x_3 & - & x_4 & + & s_1 & = & 3 \\ & & x_1 & + & x_2 & - & x_3 & - & 2x_4 & + & s_2 & = & -2 \\ & & x_1 & - & 2x_2 & - & 3x_3 & - & x_4 & + & s_3 & = & -2 \end{array}$$

$$x_1, \ldots, x_4, s_1, \ldots, s_3 \geq 0.$$

Because all the coefficients in the objective function is positive we can apply the algorithm. The first tableau:

B	x_1	x_2	x_3	x_4	sy_1	s_2	s_3	$\mathbf{x}_B$
s_1	-1	2	1	-1	1	0	0	3
s_2	1	1	-1	-2	0	1	0	-2
s_3	1	$\boxed{-1}$	-1	-1	0	0	1	-2
d_j	2	1	3	1	0	0	0	0

For the outgoing variable we choose y_3 because there are only 1 and -1 int the third row:

B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	$\mathbf{x}_B$
s_1	1	0	-1	-3	1	0	-2	-1
s_2	1	0	-2	$\boxed{-3}$	0	0	1	-4
x_2	-1	1	1	1	0	0	-1	2
d_j	3	0	2	0	0	0	1	-2

This transformation leads to the final tableau:

B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	$\mathbf{x}_B$
s_1	0	0	1	0	1	-1	-3	-3
x_4	$-\frac{1}{3}$	0	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{4}{3}$
x_2	$-\frac{2}{3}$	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
d_j	3	0	2	0	0	0	1	-2

Thus the solution is $x_2 = -\frac{2}{3}$, $x_4 = -\frac{4}{3}$, $s_1 = 3$ and $x_1 = x_3 = s_2 = s_3 = s_4 = 0$. The value of the objective function is 28.

Chapter 5

Integer and mixed integer linear programming

5.1 Exercises

1. In some model we have a variable x that is allowed to take a value from the following set $F = \{0, -1, -2, -3\}$. How can you formulate this requirement with integer programming constraint(s) and/or bounds?

   Answer:

   We can consider a couple of different solutions for this problem:

   (i) The trivial solution for the problem is to give individual bounds on x and, so $-3 \leq x \leq 0,$ integer.

   (ii) We can give a more general solution if we consider the set of feasible values we can assign a new variable for each. Let y_0, y_1, y_2 and y_3 be new binary variables ($0 \leq y_i \leq 1,$ integer; $i = 0, \ldots, 3$). If we introduce a constraint that only one of the y_i variables can be 1 (so they are mutually exclusive) we get $y_0+y_1+y_2+y_3 = 1$. This way x can be modeled as $x = 0y_0 - 1y_1 - 2y_2 - 3y_3$.

   (iii) The situation can be simplified if we omit y_0 and modify the constraint on the y_i variables: $x = -y_1 - 2y_2 - 3y_3;\ y_1 + y_2 + y_3 \leq 1;\ 0 \leq y_1, y_2, y_3 \leq 1,$ integer.

   (iv) If we are using binary variables and notice that the values are specific in the given feasibility set we can further simplify the formulation as: $x = -y_1 - 2y_2;\ 0 \leq y_1, y_2 \leq 1,$ integer.

2. Convert the following discrete optimization problem into a mixed integer linear

programming problem.

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \\ & x_1 \in \{r_1, r_2, \ldots, r_q\}. \end{aligned}$$

Note: $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$.

Answer:

The situation here is a bit similar to the previous example but there is a significant difference. Since the values are not given here we must use a general formalism to handle the feasibility set of the variable x_1. We have to set up a binary variable for each r_i value. So the solution is:

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & x_1 = r_1y_1 + r_2y_2 + \ldots + r_qy_q \\ & \mathbf{x} \geq \mathbf{0} \\ & 0 \leq y_i \leq 1, \text{ integer; } i = 1, \ldots, r \end{aligned}$$

3. A trading company is considering four investments: Investment 1 will yield a net present value (NPV) of \$16,000; investment 2, an NPV of \$22,000; investment 3, an NPV of \$12,000; and investment 4, an NPV of \$8,000. Each investment requires a certain cash outflow at the present time: investment 1, \$5,000; investment 2, \$7,000; investment 3, \$4,000; and investment 4, \$3,000. Currently, \$14,000 is available for investment.

   (a) Formulate an IP whose solution will tell the company how to maximize the NPV obtained from investments 1–4.

   (b) Modify the formulation to account for each of the following requirements:

      (*i*) The company can invest in at most two investments.

      (*ii*) Investment 2 can be carried out only if investment 1 is done.

      (*iii*) If investment 2 is selected, they can't invest in investment 4.

   Answer:

   To give the maximal NPV that can be achieved the objective function must contain the result of the investments. The budget gives a constraint on the investments and each investment (or a decision) can be modeled using a binary variable. Thus we have:

$$\begin{array}{rl} \max & 16000x_1 + 22000x_2 + 12000x_3 + 80000x_4 \\ \text{s.t.} & 5000x_1 + 7000x_2 + 4000x_3 + 3000x_4 \leq 14000 \\ & 0 \leq x_i \leq 1, integer;\ i = 1, \ldots, 4 \end{array}$$

Each of the modifications (a)-(c) give new constraints for the problem:

(i) $x_1 + x_2 + x_3 + x_4 \leq 2$,

(ii) $x_1 \geq x_2$,

(iii) $x_2 + x_4 \leq 1$.

4. The following euro coins are available: $1, 2, 5, 10, 20, 50$ cents and $1, 2$ euros. Write a mathematical model to find the minimum number of coins needed to pay a given quantity q expressed in euros.

   Answer:

   A variable is assigned to each of the coin types whose sum we want to minimize:

$$\begin{array}{rl} \min & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \\ \text{s.t.} & 1x_1 + 2x_2 + 5x_3 + 10x_4 + 20x_5 + 50x_6 + 100x_7 + 200x_8 = q \\ & x_i \geq 0,\ \text{integer};\ i = 1, \ldots, 8 \end{array}$$

5. A furniture company is capable of manufacturing three types of furniture: chair, desk, and cabinets. The manufacturing of each type of furniture requires to have the appropriate type of production line available. The line needed to manufacture each type of furniture must be rented at the following rates: chair line, \$200 per week; desk line, \$150 per week; cabinet line, \$100 per week. The chair line can produce a maximum of 40 chairs per week, the desk line can produce a maximum of 53 desks per week, and the cabinet line can produce a maximum of 25 cabinets per week. The manufacture of each type of furniture also requires some amount of wood and labor as shown below. Each week, 150 hours of labor and 160 m^2 of wood are available. The variable unit cost and selling price for each type of furniture are also give. Formulate an IP whose solution will maximize the company's weekly profits.

Furniture Type	Labor (Hours)	Wood (m^2)	Sales Price (\$)	Variable Cost (\$)
Chair	2	3	8	4
Desk	3	4	12	6
Cabinet	6	4	15	8

Answer:

First, we assign three variables x_1, x_2, x_3 to the produced quantities of chairs, desks and cabinets, respectively. Furthermore three more variables must express the usage of the production lines (y_1, y_2, y_3). The objective is to maximize the total profit, which can be obtained from the sales prices the variable costs and the line rates. Constraints on the labor and wood capacity are trivial and also the model must guarantee that production can be done only if the corresponding line is rented. These constraints use the upper bounds on the produced quantities. So the IP model is:

$$
\begin{aligned}
\max \quad & (8-4)x_1 + (12-6)x_2 + (15-8)x_3 - 200y_1 - 150y_2 - 100y_3 \\
\text{s.t.} \quad & 2x_1 + 3x_2 + 6x_3 \leq 150 \\
& 3x_1 + 4x_2 + 4x_3 \leq 160 \\
& x_1 \leq 40y_1 \\
& x_2 \leq 53y_2 \\
& x_3 \leq 25y_3 \\
& x_i \geq 0,\ \text{integer};\ 0 \leq y_i \leq 1,\ \text{integer};\ i = 1, \ldots, 3
\end{aligned}
$$

6. You have 5 keys and 6 locks. Every key opens one or more locks as shown in the following table:

	Key1	Key2	Key3	Key4	Key5
Lock1	x	x	x		
Lock2			x	x	
Lock3	x				x
Lock4	x	x			x
Lock5		x		x	
Lock6		x		x	x

Write an optimization model that chooses the minimum number of keys such that any of the locks can be opened.

Answer:

A binary variable must be assigned to each key and we want to minimize the sum of these variables. The working logic of the keys can be modeled in the constraints, each lock is represented by a constraint if at least one of the applicable keys is chosen. The IP model is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{5} x_i \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \geq 1 \\
& x_3 + x_4 \geq 1 \\
& x_1 + x_5 \geq 1 \\
& x_1 + x_2 + x_5 \geq 1 \\
& x_2 + x_4 \geq 1 \\
& x_2 + x_4 + x_5 \geq 1 \\
& 0 \leq x_i \leq 1, \ \text{integer}; \ i = 1, \ldots, 5
\end{aligned}
$$

7. Company wants to build plants to supply customers. There are m customers and n potential locations for plants.

Problem data:

n	potential locations for plants
m	number of customers
c_{ij}	cost of supplying one unit of demand i from plant j
f_j	fixed cost of opening (building) a plant in location j
d_i	demand of customer i
s_j	supply available at plant j (if open)

Decision variables:

x_j^i	units of product delivered from plant j to customer i
y_j	binary variable: $= 1$ if plant j is to be built, 0 otherwise.

Formulate a mixed integer LP problem to minimize costs.

Answer: The objective is to minimize cost while satisfying every customer demand and ensuring that a plant is built if delivery is planned from this site. The MILP model for the problem is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{j=1}^{n} f_j y_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = d_i,\ i = 1,\ldots,m \\
& \sum_{i=1}^{n} x_{ij} - s_j\delta_j \leq 0,\ j = 1,\ldots,n \\
& x_{ij} \geq 0,\ i = 1,\ldots,m,\ j = 1,\ldots,n \\
& 0 \leq \delta_j \leq 1,\ \text{integer},\ j = 1,\ldots,n.
\end{aligned}
$$

8. A car manufacturing company is considering the production of three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by each type of car are given in the table. Currently, 6,000 tons of steel and 60,000 hours of labor are available. For production of a type of a car to be economically feasible, at least 1,000 cars of that type must be produced. Formulate an IP model to maximize the company's profit.

	Car Type		
Resource	Compact	Midsize	Large
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit ($)	2,000	3,000	4,000

Answer:

Decision variables x_i correspond to the produced quantities of the different types of cars. The constraints on the available material and manpower are trivial. Also, the objective function can be modeled easily. To model that the production is economically feasible, we must introduce a new binary variable to denote whether the produced quantities are zero or not and we can use it to give a lower bound on the production if necessary. We can use one of the given constraints (e.g., the steel constraint) to compute the upper bounds on the x_i variables. Thus our model is:

$$\begin{aligned}
\max \quad & 2000x_1 + 3000x_2 + 4000x_3 \\
\text{s.t.} \quad & 1.5x_1 + 3x_2 + 5x_3 \leq 6000 \\
& 30x_1 + 25x_2 + 40x_3 \leq 60000 \\
& 1000y_1 \leq x_1 \leq 3000y_1 \\
& 1000y_2 \leq x_2 \leq 2000y_2 \\
& 1000y_3 \leq x_3 \leq 1200y_3 \\
& x_i \geq 0, \text{ integer; } 0 \leq y_i \leq 1, \text{ integer; } i = 1, \dots, 3
\end{aligned}$$

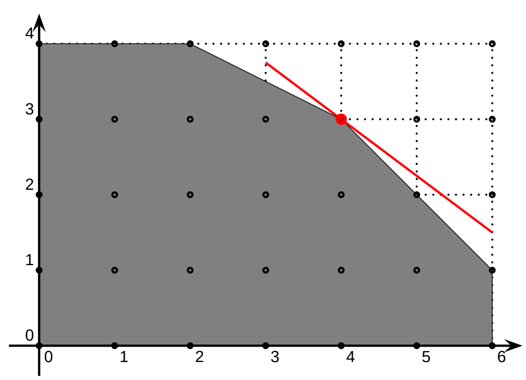
9. Solve the following two dimensional mixed integer linear programming problem graphically using your own drawing in a graph similar to the one below. It need not be very accurate. If in doubt, rely on the given numerical data.

   The objective is to maximize $z = -x_1 + 2x_2$, where x_1 is a general nonnegative integer, x_2 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $P_1(0,0)$, $P_2(0,1)$, $P_3(1,3)$, $P_4(3,4)$, $P_5(4,3)$ and $P_6(2,0)$. Where are the feasible solutions of the problem located? Determine an optimal solution. Is it unique? If not, can you find them all? How many are there? Compare the situation with continuous LP.

   Answer:

   The feasible region is shown in the figure below. Since the variable x_1 is integer, the feasible solutions are show in green. The objective function and the optimal solutions are highlighted in red. The optimal solution is not unique, there are three of them. The objective value is 5.

   If we drop the integer constraint (continuous case) the feasible solutions are covered by the polygon itself and there are infinitely many optimal solutions on the red line.



10. Find graphically the feasible region of the following integer linear programming

problem.

$$\begin{array}{lrcrcr} \min & -3x & - & 4y & & \\ \text{s.t.} & x & + & 2y & \leq & 10 \\ & x & + & y & \leq & 7 \\ & \multicolumn{5}{c}{0 \leq x \leq 6,\ 0 \leq y \leq 4} \\ & \multicolumn{5}{c}{x, y \text{ integer}} \end{array}$$

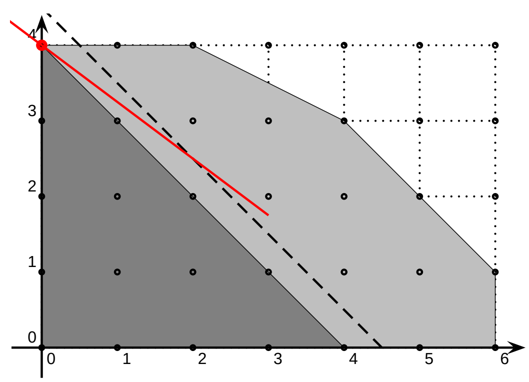
(*i*) Can you visually identify the optimal solution of this problem?

(*ii*) What is the optimal solution if the $2x + 2y \leq 9$ additional constraint is also imposed on the LP?

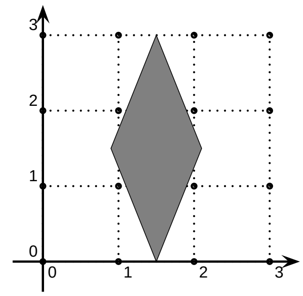
Answer:

The feasible region is shown in gray in the figure below. This region can be obtained by drawing the lines determined by the constraints. The vertices of the feasible polygon are: $(0, 0), (0, 4), (2, 4), (4, 3), (6, 1), (6, 0)$.

(*i*) First, determine the slope of the objective function. Then move it up so that it touches the feasible region in an integer point. This is the red dot in the figure below.



(*ii*) The optimal solution changes if we add the new constraint given in (*ii*), represented by the dashed line. It also can be obtained visually. We have to consider that we are interested only in integer solutions that satisfy the new constraint within the given polygon. The solution is, again, denoted by a red dot.

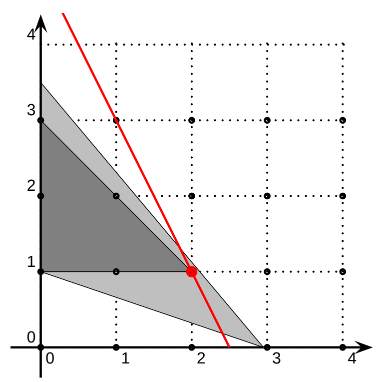


11. Solve the following integer programming problem graphically:

$$
\begin{array}{lrcrcr}
\max & x & + & y & & \\
\text{s.t.} & -10x & + & 4y & \leq & -3.0 \\
& 2.5x & + & y & \leq & 6.75 \\
& 5x & - & 2y & \leq & 7.5 \\
& 2.5x & + & y & \geq & 3.75 \\
& \multicolumn{5}{l}{0 \leq x, y \leq 3 \text{ and integer.}}
\end{array}
$$

Answer:

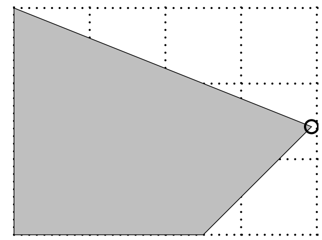
If we draw the feasible region of the problem, we obtain a polygon that does not contain integer solutions, thus the problem is integer infeasible:



12. The objective is to maximize $z = 2x_1 + x_2$, where x_2 is a general nonnegative integer, x_1 is nonnegative. The feasible region of the LP relaxation of the problem is determined by the polygon with vertices: $P_1(0, 1)$, $P_2(0, 3.5)$ and $P_3(2.95, 0)$. Where are the feasible solutions located? Determine an optimal solution graphically. Is it unique?

Answer:

The feasible solutions of the problem are located within the given polygon. They correspond to the convex hull of the integer points within the polygon. The optimal solution is uniquely determined, the optimum value is 5.



Chapter 6

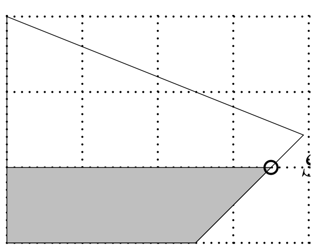
Branch-and-bound techniques, cutting plane algorithms

1. Solve the following integer programming problem using the B&B method.

$$
\begin{array}{rrcrcr}
\min z = & -3x_1 & - & 4x_2 & + & 20 \\
\text{s.t.} & \frac{2}{5}x_1 & + & x_2 & \leq & 3 \\
 & \frac{2}{5}x_1 & - & \frac{2}{5}x_2 & \leq & 1 \\
 & \multicolumn{5}{c}{x_1, x_2 \geq 0 \text{ and integer.}}
\end{array}
$$

Answer:

P_0 : LP relaxation

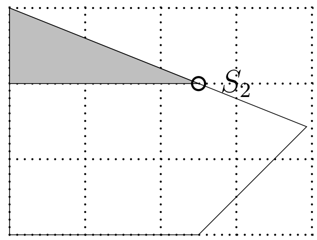


At S_0 : $x_1 = 3\frac{13}{14},\ x_2 = 1\frac{3}{7};\quad z_0 = 2\frac{1}{2}$

Both variables have fractional value. Branch on $x_2 \Rightarrow P_1 := P_0 \,\&\, x_2 \leq 1;\quad P_2 := P_0 \,\&\, x_2 \geq 2$. Branching on x_1 could also have been performed.

Examine waiting (pending) nodes: $W = \{P_1, P_2\}$.

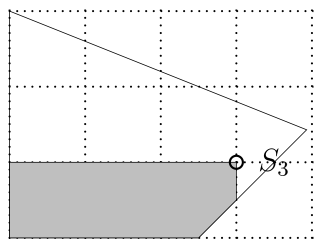
First investigate $P_1 := P_0 \,\&\, x_2 \leq 1$ by solving its LP relaxation:



At S_1 : $x_1 = 3\frac{1}{2},\ x_2 = 1; \quad z_1 = 5\frac{1}{2}$

The LP relaxation is not integer feasible as x_1 is not integer.

Next, investigate $P_2 := P_0\ \&\ x_2 \geq 2$:



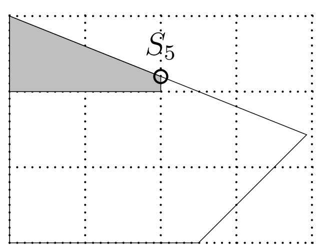
At S_2 : $x_1 = 2\frac{1}{2},\ x_2 = 2; \quad z_2 = 4\frac{1}{2}$.

Again, the LP relaxation is not integer feasible as x_1 is not integer.

From among the two pending nodes choose P_1 and branch on x_1 and generate $P_3 := P_1\ \&\ x_1 \leq 3$ and $P_4 := P_1\ \&\ x_1 \geq 4$.

Now we have three waiting nodes: $W = \{P_2, P_3, P_4\}$.

Investigate $P_3 := P_1\ \&\ x_1 \leq 3$:



At S_3 : $x_1 = 3,\ x_2 = 1$; and $z_3 = 7$.

Solution is integer feasible with $Z = z_3 = 7$.

The remaining waiting nodes: $W = \{P_2, P_4\}$.

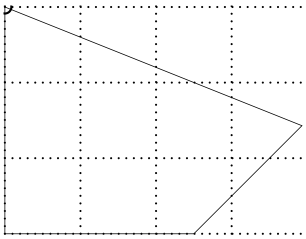
Choose $P_4 := P_1\ \&\ x_1 \geq 4$:

No feasible solution because x_1 cannot be greater than 3.93 (see P_0).

Waiting node: $W = \{P_2\}$.

Take P_2 and branch on x_1. It results in $P_5 := P_2\ \&\ x_1 \leq 2$ and $P_6 := P_2\ \&\ x_1 \geq 3$. Waiting nodes: $W = \{P_5, P_6\}$.

Investigate $P_5 := P_2\ \&\ x_1 \leq 2$ first:



At $S_5 : x_1 = 2,\ x_2 = 2\frac{1}{5}$; and $z_5 = 5\frac{1}{5}$.

This node is pending as in the solution x_2 is not integer.

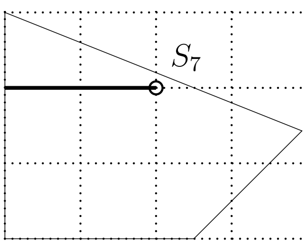
Waiting nodes: $W = \{P_5, P_6\}$.

Take $P_6 := P_2$ & $x_1 \geq 3$:

No feasible solution because x_1 cannot be greater than 3 if $x_2 \geq 2$. (The LP solver would notice this infeasibility of P_6.)

Choose P_5, branch on x_2 resulting in $P_7 := P_5$ & $x_2 \leq 2$ and $P_8 := P_5$ & $x_2 \geq 3$. Waiting nodes: $W = \{P_7, P_8\}$.

Select $P_8 := P_5$ & $x_2 \geq 3$:



At $S_8 :\quad x_1 = 0,\ x_2 = 3;\quad z_8 = 8.$

Integer solution but worse than $Z = z_3 = 7$.

Remaining waiting node: $W = \{P_7\}$.

Select $P_7 := P_5$ & $x_2 \leq 2$:

At $S_7 : x_1 = 2,\ x_2 = 2$; and $z_7 = 6$.

Integer solution, better than Z. Therefore, $Z := z_7 = 6$.

No more pending nodes: z_7 **is an optimal solution!**

Chapter 7

Network optimization

7.1 Exercises

1. Give the mathematical model of the transportation problem.

   Answer:

   Let $\mathcal{M}$ and $\mathcal{N}$ be two sets:

   - $\mathcal{M}$: set of supply nodes
   - $\mathcal{N}$: set of demand nodes

   Let the parameters be the following:

   - m: number of supply nodes
   - n: number of demand nodes
   - s_i: supply at node i ($i = 1, \ldots, m$)
   - d_j: demand at node j ($j = 1, \ldots, n$)
   - c_{ij}: cost of transporting one unit from supply node i to demand node j

   Variable:

   - x_{ij}: quantity transported from node i to node j

   Assumption: Total supply is equal to total demand.

   Objective: To satisfy every demand at minimal total cost.

$$min \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{N}} c_{ij} x_{ij}$$

$$\sum_{i \in \mathcal{M}} x_{ij} = d_j, \;\; j = 1, \ldots, n$$

$$\sum_{j \in \mathcal{N}} x_{ij} = s_i, \;\; i = 1, \dots, m$$

$$\sum_{i \in \mathcal{M}} s_i = \sum_{j \in \mathcal{N}} d_j$$

$$x_{ij} \geq 0, \;\; \forall (i,j)$$

2. Find a starting basis of the following transportation problem using the North-West Corner Rule Method. Is the starting basis optimal?

$$\mathbf{s} = \begin{pmatrix} 4 \\ 7 \\ 2 \end{pmatrix}, \qquad \mathbf{d} = (3, 5, 4, 1), \qquad \mathbf{C} = \begin{pmatrix} 4 & 2 & 5 & 6 \\ 4 & 1 & 3 & 7 \\ 8 & 6 & 5 & 4 \end{pmatrix}$$

Answer:

The total supply and the total demand is equal to 13. We have to solve the augmented problem:

Cost coefficients

				Supply
4	2	5	6	4
4	1	3	7	7
8	6	5	4	2
3	5	4	1	
Demand				

For a starting basis, the North-West corner rule is used. This basis is:

4 **3**	2 **1**	5	6	4
4	1 **4**	3 **3**	7	7
8	6	5 **1**	4 **1**	2
3	5	4	1	

The starting basis is optimal, because the reduced costs of nonbasic cells are non negative.

3. Find a starting basis of the following transportation problem using the Least Cost Cell Method. Is this basis optimal?

$$\mathbf{s} = \begin{pmatrix} 20 \\ 12 \\ 30 \end{pmatrix}, \qquad \mathbf{d} = (15, 20, 15, 12), \qquad \mathbf{C} = \begin{pmatrix} 9 & 7 & 6 & 6 \\ 8 & 6 & 7 & 9 \\ 7 & 8 & 8 & 5 \end{pmatrix}$$

Answer:

The starting basis using the Least Cell Cost Method is the following:

9	7 **5**	6 **15**	6	20
8	6 **12**	7	9	12
7 **15**	8 **3**	8	5 **12**	30
15	20	15	12	

The starting basis is optimal, because the reduced costs of nonbasic cells are non negative.

4. Solve the following transportation problem using the North-West Corner Rule Method.

$$\mathbf{s} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}, \qquad \mathbf{d} = (2, 4, 2, 2, 2), \qquad \mathbf{C} = \begin{pmatrix} 2 & 3 & 4 & 1 & 2 \\ 4 & 5 & 3 & 2 & 1 \\ 1 & 3 & 4 & 6 & 2 \end{pmatrix}$$

Answer:

An optimal solution is:

2	3 **1**	4	1 **2**	2	3
4	5 **1**	3 **2**	2	1 **2**	5
1 **2**	3 **2**	4	6	2	4
2	4	2	2	2	

Objective value = 26

5. Solve the following transportation problem using the Least Cost Cell Method.

$$\mathbf{s} = \begin{pmatrix} 10 \\ 7 \\ 12 \\ 11 \end{pmatrix}, \qquad \mathbf{d} = (10, 10, 10, 10, 10, 10), \qquad \mathbf{C} = \begin{pmatrix} 2 & 6 & 5 & 3 \\ 4 & 3 & 4 & 2 \\ 2 & 6 & 4 & 4 \\ 6 & 8 & 7 & 9 \end{pmatrix}$$

This basis is optimal because all

6. Solve the following transportation problem using the North-West Corner Rule Method.

$$\mathbf{s} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix}, \qquad \mathbf{d} = (4, 4, 4, 4, 4, 4), \qquad \mathbf{C} = \begin{pmatrix} 4 & 3 & 2 & 5 & 7 & 2 \\ 3 & 4 & 3 & 5 & 3 & 7 \\ 6 & 5 & 4 & 3 & 4 & 2 \end{pmatrix}$$

7. Solve the following transportation problem using the Least Cost Cell Method.

$$\mathbf{s} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix}, \qquad \mathbf{d} = (4,4,4,4,4,4), \qquad \mathbf{C} = \begin{pmatrix} 4 & 3 & 2 & 5 & 7 & 2 \\ 3 & 4 & 3 & 5 & 3 & 7 \\ 6 & 5 & 4 & 3 & 4 & 2 \end{pmatrix}$$

8. Solve the following transportation problem, where $x_{11} = x_{12} = x_{23} = x_{34} = 0$ (x_{ij} is the quantity transported from node i to node j).

$$\mathbf{s} = \begin{pmatrix} 7 \\ 6 \\ 8 \end{pmatrix}, \qquad \mathbf{d} = (5,3,5,5,3), \qquad \mathbf{C} = \begin{pmatrix} 2 & 3 & 4 & 2 & 5 \\ 3 & 3 & 1 & 4 & 3 \\ 2 & 2 & 4 & 3 & 4 \end{pmatrix}$$

9. Suppose that a taxi firm has four taxis available, and four customers wishing to be picked up as soon as possible. The firm prides itself on speedy pickups, so for each taxi the "cost" of picking up a particular customer will depend on the time taken for the taxi to reach the pickup point (see the "cost" matrix $\mathbf{C}$, where c_{ij} defines the distance in time between the taxi i and customer j). Give an optimal "taxi-customer" assignment where the total waiting time of the customers is minimal.

$$\mathbf{C} = \begin{pmatrix} 14 & 5 & 8 & 7 \\ 2 & 12 & 6 & 5 \\ 7 & 8 & 3 & 9 \\ 2 & 4 & 6 & 10 \end{pmatrix}$$

Chapter 8

Game theory

8.1 Exercises

1. A 2p0sg has the following reward matrix:

	C's strategy		
R's strategy	C1	C2	C3
R1	17	23	48
R2	17	3	51
R3	6	17	3

Which strategy should each of the two players choose? One answer must be obtained by applying the concept of dominated strategies to rule out a succession of inferior strategies until only one choice remains.

First, apply the notion of dominance.

At the initial table (reward matrix) there are no dominated strategies for **C**. However, for **R**, strategy R3 is dominated by R1 because the latter has larger payoffs regardless of what **C** does. Eliminating strategy R3 from further consideration the following reward matrix is obtained:

	C's strategy		
R's strategy	C1	C2	C3
R1	17	23	48
R2	17	3	51

C now has a dominated strategy which is C3. It is dominated by both strategies C1 and C2 because they always have smaller losses. Eliminating this strategy we obtain the following reward matrix:

	C's strat	
R's strategy	C1	C2
R1	17	23
R2	17	3

Now strategy R2 for **R** becomes (weakly) dominated by strategy R1 for. Eliminating the dominated strategy the following table is obtained:

	C's strat	
R's strategy	C1	C2
R1	17	23

Strategy C2 for **C** is dominated by C1, as $17 < 23$. Consequently, both players should choose strategy 1.

If we follow the steps of finding a saddle point, where max of row minima is equal to min of col maxima, we notice that in this game saddle point does exist and it is achieved if both players choose their respective first strategies. The value of the game is 17.

2. Three linear functions $y_1,\ y_2$ and y_3 are defined as follows:

$$\begin{array}{rcrcr} y_1 & = & 2 & - & x \\ y_2 & = & x & - & 1 \\ y_3 & = & 2x & - & 6 \end{array}$$

$$\text{Find } \min_{x \geq 0} \max_i \{y_i\}.$$

One possibility to solve the problem is to introduce an auxiliary variable, say x_0 and then to solve the equivalent problem with $x \geq 0$:

$$\begin{array}{llcrl} \min & x_0 & & & \\ \text{s.t.} & 2 & - & x & \leq x_0 \\ & x & - & 1 & \leq x_0 \\ & 2x & - & 6 & \leq x_0 \end{array}$$

After rearranging:

$$\begin{array}{llcrcr} \min & x_0 & & & & \\ \text{s.t.} & x_0 & + & x & \geq & 2 \\ & x_0 & - & x & \geq & -1 \\ & x_0 & - & 2x & \geq & -6 \\ & & & x & \geq & 0 \end{array}$$

Optimal solution is: $x = 1.5,\ x_0 = 0.5.$

Problem can also be solved graphically. Try it.

3. The manager of a multinational company and the union of workers are preparing to sit down at the bargaining table to work out the details of a new contract for the workers. Each side has developed certain proposals for the contents of the new contract. Let us call union proposals “Prop-1”, “Prop-2” and “Prop-3, and the manager proposals “Contr-A” (for contract), “Contr-B” and “Contr-C”. Both parties are aware of the financial consequences of each proposalontract combination. The pay-off matrix is:

	Manager’s		
Workers’	Contr-A	Contr-B	Contr-C
Prop-1	8.5	7.0	7.5
Prop-2	12.0	9.5	9.0
Prop-3	9.0	11.0	8.0

These values are the contract gains that the workers’ union would secure and also the cost the company would have to bear.

Is there a clearut contract combination agreeable to both parties, or will they find it necessary to submit to arbitration in order to arrive at some sort of compromise?

The workers’ union is the row player and the manager is the column player. We have to check if this game has an equilibrium point.

First, determine union’s optimal strategy: compute row minima.

	Manager’s			
Workers’	Contr-A	Contr-B	Contr-C	Row minimum
Prop-1	8.5	7.0	7.5	7.0
Prop-2	12.0	9.5	9.0	9.0
Prop-3	9.0	11.0	8.0	8.0

As the largest row minimum is at proposal-2 the union will select this strategy.

In a similar way we find the manager optimal strategy. The maximum “loss” of the manager for each strategy of the union is shown in the “Col max” row of the following table. The minimum of these maximum pay-outs is 9.0 in the third column. Consequently the manager would select Contr-C as his optimal one.

	Manager’s			
Workers’	Contr-A	Contr-B	Contr-C	Row minimum
Prop-1	8.5	7.0	7.5	7.0
Prop-2	12.0	9.5	9.0	9.0
Prop-3	9.0	11.0	8.0	8.0
Col max	12.0	11.0	9.0	

In this game both of the players will select a strategy that has the same value. The min value in row 2 is also the max value in column C, so the solution is an equilibrium or saddle point.

$$\min\{col\ max\} = \min\{12, 11, 9\} = 9 = \max\{row\ min\} = \max\{7, 9, 8\}.$$

The two sides can reach an agreement. There is no need for arbitration.

4. Consider the same situation as in Problem 3, but with the following pay-off matrix:

	Manager's		
Workers'	Contr-A	Contr-B	Contr-C
Prop-1	9.5	12.0	7.0
Prop-2	7.0	8.5	6.5
Prop-3	6.0	9.0	10.0

Is there an equilibrium point?

Find the mixed strategies for the union and the manager.

Formulate (but do not solve) the LP problem to determine the optimum strategy for the union and the optimum strategy of the manager.

First, determine row minima and column maxima.

	Manager's			
Workers'	Contr-A	Contr-B	Contr-C	Row minimum
Prop-1	9.5	12.0	7.0	7.0
Prop-2	7.0	8.5	6.5	6.5
Prop-3	6.0	9.0	10.0	6.0
Col max	9.5	12.0	10.0	

There is no equilibrium point as max row min = 7.0 and it is not equal to min col max = 9.5. Randomized strategies have to be used.

Let x_1, x_2, x_3 denote the (currently unknown) probabilities of the union where x_i represents the probability that union chooses the i-th proposal. The manager's probabilities of choosing the j-th contract are denoted by y_1, y_2, y_3.

If the union chooses the mixed strategy (x_1, x_2, x_3) then their expected reward against each of manager's strategies are:

Manager chooses	Union's reward
Contr-A	$9.5x_1 + 7x_2 + 6x_3$
Contr-B	$12x_1 + 8.5x_2 + 9x_3$
Contr-C	$7x_1 + 6.5x_2 + 10x_3$

By the basic assumption the manager will choose a strategy that makes union expected reward equal to

$$\min\{(9.5x_1 + 7x_2 + 6x_3), (12x_1 + 8.5x_2 + 9x_3), (7x_1 + 6.5x_2 + 10x_3)\} \quad (8.1)$$

and at the same time the union should choose the strategy (x_1, x_2, x_3) to make (8.1) as large as possible:

$$\max\{\min\{(9.5x_1 + 7x_2 + 6x_3), (12x_1 + 8.5x_2 + 9x_3), (7x_1 + 6.5x_2 + 10x_3)\}\}.$$

Therefore the union's strategy is the solution of the following LP:

$$\begin{array}{lrcrcrcrcl}
\max & v & & & & & & & & \\
\text{s.t.} & v & - & 9.5x_1 & - & 7.0x_2 & - & 6.0x_3 & \leq & 0 \\
& v & - & 12.0x_1 & - & 8.5x_2 & - & 9.0x_3 & \leq & 0 \\
& v & - & 7.0x_1 & - & 6.5x_2 & - & 10.0x_3 & \leq & 0 \\
& & & x_1 & + & x_2 & + & x_3 & = & 1 \\
& & & & & & & x_1,\ x_2,\ x_3 & \geq & 0
\end{array}$$

In a similar fashion the manager strategy will be determined by the solution of the following LP problem:

$$\begin{array}{lrcrcrcrcl}
\min & w & & & & & & & & \\
\text{s.t.} & w & - & 9.5y_1 & - & 12.0y_2 & - & 7.0y_3 & \geq & 0 \\
& w & - & 7.0y_1 & - & 8.5y_2 & - & 6.5y_3 & \geq & 0 \\
& w & - & 6.0y_1 & - & 9.0y_2 & - & 10.0y_3 & \geq & 0 \\
& & & y_1 & + & y_2 & + & y_3 & = & 1 \\
& & & & & & & y_1,\ y_2,\ y_3 & \geq & 0
\end{array}$$

Chapter 9

Nonlinear programming

1. Determine whether the following functions are convex or not for $x \in \mathbb{R}^1$:

$$f(x) = 1 + 2x + x^2, \qquad g(x) = x^2 + e^{-x}, \qquad h(x) = x^2 - e^x.$$

Answer:

$f(x)$ is convex as it can be written as $(1+x)^2$, and a quadratic function is convex.

$g(x)$ is convex as it is the sum of two convex functions.

$h(x)$ is not convex as it is dominated by $-e^x$ which is a concave function. $h(x)$ is the sum of a convex and a concave function.

2. Determine whether the following functions are convex for $\mathbf{x} > \mathbf{0}$. Note, $\mathbf{x} = (x_1, x_2)$.

$f(x_1, x_2) = 4x_1^2 - 4x_1x_2 + x_2^2 - \log(x_1), \quad g(x_1, x_2) = 4x_1^2 + x_2^2 + 4x_1x_2 + \log(x_1x_2).$

Answer:

f can be written as $(2x_1 - x_2)^2 - \log(x_1)$ which is the sum of two convex functions ($-\log(x_1)$ is convex). Therefore, f is convex.

g can be written as $(2x_1 + x_2)^2 + \log(x_1x_2)$ which is the sum of a convex and a concave function, thus g is not convex.

3. Show that the following function is convex and determine its minimum

$$f(x) = \frac{11}{273}x^6 - \frac{19}{91}x^4 + x^2.$$

Answer:

$f^{''}(x) = \dfrac{330}{273}x^4 - \dfrac{228}{91}x^2 + 2$. First show that $f^{''}(x)$ is always positive (thus f is convex). It can be done by substituting $y = x^2$ and noticing that the discriminant of the resulting quadratic equation is negative, thus there is no solution, the x axis is never crossed. Having done so, solve $f^{'}(x) = 0$ which gives $x = 0$ and $f(x) = 0$.

4. A furniture company makes wall cabinets. There is a fixed cost of production per month of €6000. The cost of making a chair is €30. Sales price affects the quantities sold:

$$\text{volume}(v) = 500 - 1.4\,\text{price}(p).$$

Work out a profit function and determine the price that will maximize profit. Also, compute the optimum value.

Answer:

Profit as a function of sales price: $f(p) = (500 - 1.4p)p - 30(500 - 1.4p) - 6000$, which simplifies to $f(p) = -1.4p^2 + 542p - 21000$. It has maximum at $p \approx 193$. The optimum value (max profit) $\approx$ €31457.

5. Find the extreme points of $f(x) = x^4 - 2x^2 + 2$. Determine whether they are local or global minima/maxima. Having done so, determine the minimum of the same function $f(x)$ subject to $-0.5 \leq x \leq 1.5$.

Answer:

Note, $f(x)$ is neither convex nor concave.

Taking the first derivative of $f(x)$ and setting it equal to zero: $f'(x) = 4x^3 - 4x = 0$, or $x(x^2 - 1) = 0$, from which $x_1 = 0,\ x_2 = -1,\ x_3 = 1$. x_1 is a local maximizer (the first derivative changes sign here from $+$ to $-$, i.e., $f(x)$ is locally concave), x_2 and x_3 are local minimizers (analogous arguments). At the same time, they are global minimizers since $f(x) \to +\infty$ as $x \to \pm\infty$ (multiple optimum with multiplicity of 2).

One local minimum of $f(x)$ falls inside the $-0.5 \leq x \leq 1.5$ feasible region. Therefore, the minimum of $f(x)$ s.t. $-0.5 \leq x \leq 1.5$ is $x = 1$ and the optimum is $f(1) = 1$.

6. Find the minimum of $g(x) = x^2 + e^{-x}$.

Answer:

Setting the first derivative of g equal to 0 the resulting equation is $2x - e^{-x} = 0$, from which $x \approx 0.35$. Since g is convex this point is a minimizer.

7. Solve the following nonlinear programming problem.

$$\min\ f(x) = \frac{1}{4}x^2 + x + 1, \quad \text{subject to } -1 \leq x \leq 2.$$

Answer:

$f(x)$ can be written as $\left(\frac{1}{2}x + 1\right)^2$ which is a convex function. It reaches its global minimum at $x = -2$. However, it falls outside the $-1 \leq x \leq 2$ feasible region.

Therefore, the minimum is attained at the boundary of the region. This is the point nearest to $x = -2$ which is $x = -1$. Thus, it is the minimum point. The optimal value is $f(-1) = 0.25$.

8. Determine which of the following functions is smooth/nonsmooth on the given domain. [Note: $f(\mathbf{x}) = f(x_1, x_2)$.]

   (i) $f(\mathbf{x}) = \log(x_1 x_2) - (x_1 + x_2)^2, \quad 0 < x_1, x_2 \leq 100$

   (ii) $g(\mathbf{x}) = |x_1 - 2| + x_2^3, \quad 0 \leq x_1, x_2 \leq +\infty$

   (iii) $h(\mathbf{x}) = |x_1 + x_2|^2, \quad -\infty \leq x_1, x_2 \leq +\infty$

   Answer:

   (i) $f(\mathbf{x})$ smooth.

   (ii) $g(\mathbf{x})$ nonsmooth, because of the absolute value term.

   (iii) $h(\mathbf{x}) = |x_1 + x_2|^2 \equiv (x_1 + x_2)^2$, which is smooth.

9. Which of the following functions have local extreme points (minimum or maximum), and if so, where? Why? [Note: $f(\mathbf{x}) = f(x_1, x_2)$.]

   (i) $f(\mathbf{x}) = 1 - x_1 x_2$

   (ii) $f(\mathbf{x}) = x_1^2 - x_2^3$

   (iii) $f(\mathbf{x}) = x_1^2 + x_2^2$

   Answer:

   Reminder: If all eigenvalues of a matrix are positive then the matrix is positive definite.

   The necessary condition for a point to be extreme for $f(\mathbf{x})$ is that the gradient vanishes in a point, $\nabla f(\mathbf{x}_0) = \mathbf{0}^T$. If in this point the Hessian is definite then this is an extreme point (minimum or maximum).

   (i) $\nabla f = [-x_2, -x_1] = \mathbf{0}$ iff $x_1 = x_2 = 0$. The Hessian:

   $$\mathbf{H} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

   The eigenvalues come from $\det[\mathbf{H} - \lambda \mathbf{I}_2] = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$, giving $\lambda_1 = -1$ and $\lambda_2 = 1$. So, the Hessian is indefinite, no local extremum.

   (ii) $\nabla f = [2x_1, -3x_2^2] = \mathbf{0}$ iff $x_1 = x_2 = 0$. The Hessian:

   $$\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & -6y \end{bmatrix}.$$

In $[0, 0]$ the eigenvalues come from det$[\mathbf{H} - \lambda\mathbf{I}_2] = (2 - \lambda)(-\lambda) = 0$, giving $\lambda_1 = 2$ and $\lambda_2 = 0$. So, the Hessian is positive semidefinite, no local extremum.

(iii) $\nabla f = [2x_1, 2x_2] = \mathbf{0}$ iff $x_1 = x_2 = 0$. The Hessian:

$$\mathbf{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is positive definite, therefore $[0, 0]$ is a local minimum.

10. Write the KKT conditions for the following problem:

$$\begin{aligned} \min\ f(\mathbf{x}) &= x_1^4 + 2x_1^2 + 2x_1x_2 + 4x_2^2 \\ s.t. \quad & 2x_1 + x_2 = 10 \\ & x_1 + 2x_2 \geq 10 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Answer:

$$\begin{aligned} 4x_1^3 + 4x_1 + 2x_2 + 2\lambda - \mu_1 - \mu_2 &= 0 \\ 2x_1 + 8x_2 + \lambda - 2\mu_1 - \mu_3 &= 0 \\ 2x_1 + x_2 - 10 &= 0 \\ x_1 + 2x_2 - 10 &\geq 0 \\ x_1, x_2 &\geq 0 \\ \mu_1(x_1 + 2x_2 - 10) &= 0 \\ \mu_2 x_1 &= 0 \\ \mu_3 x_2 &= 0, \\ \mu_1, \mu_2, \mu_3 &\geq 0 \end{aligned}$$

11. Consider the following constrained nonlinear programming problem:

$$\begin{aligned} \max\ f(\mathbf{x}) &= x_1x_2 + x_1x_3 + x_2x_3 \\ s.t. \quad & x_1 + x_2 + x_3 = 3. \end{aligned}$$

Define the KKT conditions for the problem. Find a solution that satisfies the conditions. Determine if it is a maximizer.

Answer:

The Lagrangian function of the problem is:

$$L(\mathbf{x}, \lambda) = x_1x_2 + x_1x_3 + x_2x_3 + \lambda(x_1 + x_2 + x_3 - 3).$$

A stationary point of L satisfies the following system of equations:

$$\begin{aligned}
\frac{\partial L(\mathbf{x},\lambda)}{\partial x_1} &= x_2 + x_3 + \lambda = 0 \\
\frac{\partial L(\mathbf{x},\lambda)}{\partial x_2} &= x_1 + x_3 + \lambda = 0 \\
\frac{\partial L(\mathbf{x},\lambda)}{\partial x_3} &= x_1 + x_2 + \lambda = 0 \\
\frac{\partial L(\mathbf{x},\lambda)}{\partial \lambda} &= x_1 + x_2 + x_3 - 3 = 0
\end{aligned}$$

This is a system of linear equations. It has a unique solution of $(x_1^*, x_2^*, x_3^*) = (1, 1, 1)$ and $\lambda^* = -2$. This is a candidate for maximum. The determinant of the Hessian is -3 (verify!), therefore f is concave and the point is a maximizer.

12. Consider the following nonlinear programming problem:

$$\begin{aligned}
\min \quad & 4(x_1 - 2)^2 + (x_2 - 1)^2 \\
s.t. \quad & 16x_1 + 6x_2 = 63.
\end{aligned}$$

Write the $\mathcal{L}$ Lagrangian function of the problem. Define the necessary condition of optimality for $\mathcal{L}$ and solve the resulting system. What is the solution if $x_1 \leq 2$ is also imposed?

Answer:

$\mathcal{L}(\mathbf{x}, \lambda) = 4(x_1 - 2)^2 + (x_2 - 1)^2 + \lambda(16x_1 + 6x_2 - 63)$

Setting partial derivatives of $\mathcal{L}$ equal to zero:

$$\begin{aligned}
8(x_1 - 2) + 16\lambda &= 0 \\
2(x_2 - 1) + 6\lambda &= 0 \\
16x_1 + 6x_2 &= 63
\end{aligned}$$

The solution is $x_1^* = 3,\ x_2^* = 2.5,\ (\lambda^* = -0.5)$. Since the objective function is convex the solution is a minimizer. Optimal value is 6.25.

If $x_1 \leq 2$ is also imposed, the solution is $x_1^* = 2,\ x_2^* \approx 5.17$. The optimal value is ≈ 17.36 (much worse than without the additional constraint of $x_1 \leq 2$).